Section 9.7. Periodic Solutions and Limit Cycles

Note. In this section we consider the autonomous system $\vec{x}' = \vec{f}(\vec{x})$ and look for periodic solutions of the form $\vec{x}(t+T) = \vec{x}(t)$. These solutions will be closed curves in the phase plane. We saw examples of this in the linear system $\vec{x}' = A\vec{x}$ where the eigenvalues of A were purely imaginary. We also saw this in the nonlinear predator-prey equations.

Note. The following example illustrates a system with periodic solutions which also shows a certain stability.

Example. Page 492 Example 1. Consider the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} y + x - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{bmatrix}.$$

Discuss the solutions.

Solution. Well, $(0,0)$ is certainly a critical point. The associated almost linear system is

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where A has eigenvalues $1 \pm i$. So $(0,0)$ is an unstable spiral point of the almost spiral linear system and therefore of the original system. Now from the original system, we have

$$x\frac{dx}{dt} = xy + x^2 - x^2(x^2+y^2)$$

$$y\frac{dy}{dt} = -xy + y^2 - y^2(x^2+y^2)$$

and so

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2+y^2) - (x^2+y^2)^2.$$

Letting $x = r\cos\theta$ and $y = r\sin\theta$ we have $r^2 = x^2+y^2$ and $r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$ and

$$r\frac{dr}{dt} = r^2 - r^4 = r^2(1-r^2). \qquad (*)$$

So, $dr/dt = 0$ when $r = 0$ and when $r = 1$ (here, we keep $r \geq 0$ in (r,θ) polar coordinates). Also, $dr/dt > 0$ if $r < 1$ and $dr/dt < 0$ if $r > 1$. Now for θ consider

$$y\frac{dx}{dt} - x\frac{dy}{dt} = r\sin\theta\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right) - r\cos\theta\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right) = -r^2\frac{d\theta}{dt}.$$

Also, as above,

$$y\frac{dx}{dt} - x\frac{dy}{dt} = x^2+y^2.$$

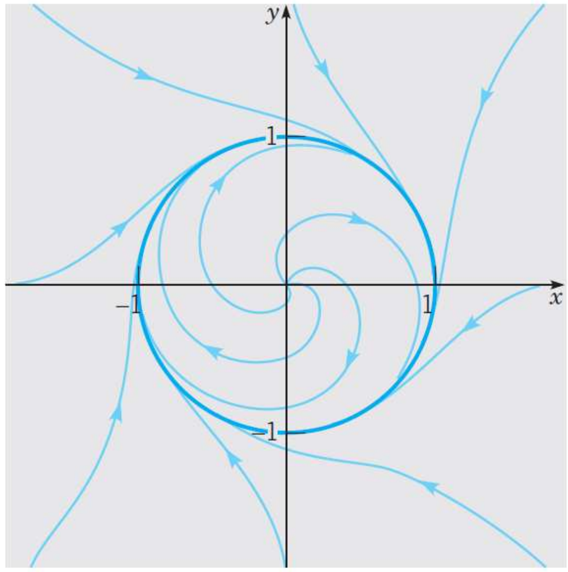
So $-r^2 d\theta/dt = r^2$ and $d\theta/dt = -1$. Solving $(*)$ we get

$$r\frac{dr}{dt} = r^1(1-r^2) \text{ or } \frac{dr}{r(1-r^2)} = dt$$

which gives

$$r = \frac{1}{\sqrt{1 + (1/\rho^2 - 1)e^{-2t}}} \text{ and } \theta = -t + \alpha$$

where $r(0) = \rho$ and $\theta(0) = \alpha$. Notice that if $\rho < 1$ then $r \to 1$ as $t \to \infty$, and if $\rho > 1$ then $r \to 1$ as $t \to \infty$. The trajectories are (this is Figure 9.7.1 in the 10 edition of DiPrima and Boyce):



Definition. A closed trajectory in the phase plane such that other trajectories spiral toward it (either from the inside or outside) as $t \to \infty$ is called a *limit cycle*.

Definition. If all the trajectories near a limit cycle (both those inside and outside) spiral towards the limit cycle as $t \to \infty$, then the limit cycle is said to be *stable*. If the trajectories on one side spiral towards and on the other side spiral away, then the limit cycle is *semistable*. If the trajectories on both sides of a closed trajectory spiral away as $t \to \infty$, then the closed trajectory is *unstable* (it's not even called a limit cycle). In the case that nearby trajectories neither approach nor depart a closed trajectory, it is called *neutrally stable*.

Note. Now rewrite $\vec{x}' = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

Theorem 9.7.1. Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. A closed trajectory must necessarily enclose at least one critical point. If it encloses only one critical point, then the critical point cannot be a saddle point.

Theorem. Pincaré Bendixson.

Let F and G have continuous first partial derivatives in a domain D of the xy-plane. Let D_1 be a bounded subdomain in D and let R be the region that consists of D_1 and its boundary. Suppose that R contains no critical point. If $x = \varphi(t)$, $y = \psi(t)$ is a solution for all $t \geq t_0$, then either:

1. $x = \varphi(t)$, $t = \psi(t)$ is a periodic solution, or

2. $x = \varphi(t)$, $y = \psi(t)$ spirals towards a closed trajectory as $t \to \infty$.

In either case, R contains a periodic solution.

Example. Page 492 Example 1. Consider (again) the system

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} y + x - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{bmatrix}.$$

Apply the Poincaré-Bendixson Theorem to show that this system has a periodic solution.

Solution. We saw above that $dr/dt = r(1-r^2)$. Now, in the region $\{r, \theta) \mid 1/2 < r < 3/2\}$, $\left.\frac{dr}{dt}\right|_{r=1/2} > 0$ and $\left.\frac{dr}{dt}\right|_{r=3/2} < 0$ so any trajectory in the region remains in the region. Hence, by the Poincaré-Bendixson Theorem, the region must contain a periodic solution.

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