Section 9.8. Chaos and Strange Attractors: The Lorentz Equations

Note. In this section we describe the behavior of a single system of three (nonlinear) DEs in three variables. It has been realized in the last several years that third and higher order systems can exhibit very complicated behavior not seen in second order systems.

Note. In the early 1960s, Edward Lorentz was running meteorological models on his computer. He was numerically solving the equations:

$$\begin{cases} \frac{dx}{dt} = \sigma(-x+y) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = -bz + xy. \end{cases}$$

This system has the following critical points (x,y,z):

$$(0,0,0),\ (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1),\ \text{and } (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

We'll call these points P_1, P_2, P_3, respectively. Notice that there is actually only one critical point if $r \leq 1$. We can write the Lorentz equations as

$$\vec{x}\,' = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix}.$$

This system is almost linear at $(0,0,0)$. For physical reasons, $\sigma = 10$, and $b = 8/3$. Considering the linear part, we find the eigenvalues

$$\lambda_1 = -\frac{8}{3},\ \lambda_2 = \frac{-11 - \sqrt{81+40r}}{2},\ \text{and } \lambda_3 = \frac{-11+\sqrt{81+40r}}{2}.$$

for $r < 1$, the eigenvalues are negative and $(0,0,0)$ is asymptotically stable ("globally"). If $r > 1$, $(0,0,0)$ is unstable.

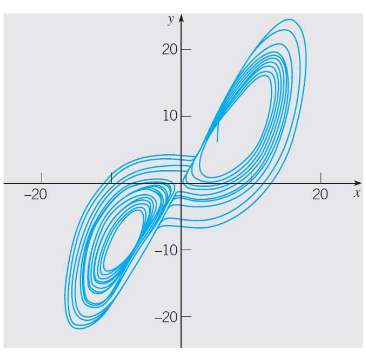
Exploring P_2 and P_3 we find (see page 504):

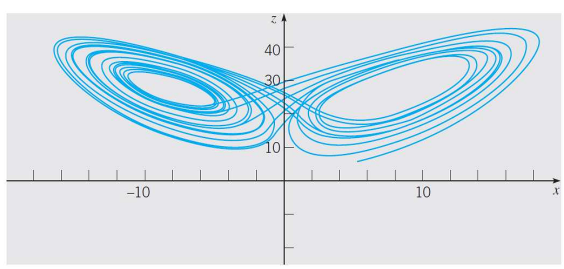
1. If $1 < r < 1.34$, then P_2 and P_3 are asymptotically stable. Solutions asymptotically approach P_2 or P_3 (not spiraling).

2. If $1.34 < r < 24.74$ then P_2 and P_3 are asymptotically stable and solutions spiral toward P_2 or P_3.

3. If $r > 24.74$ then all critical points are unstable.

Note. It can be shown (see Page 509 Number 5) that the trajectories remain bounded as $t \to \infty$. In fact, trajectories approach a limiting set of zero volume. This attracting set is called a *strange attractor* and the term *chaotic* is used to describe the trajectories. These are Figures 9.8.5 and 9.8.6 in the 10 edition of DiPrima and Boyce:





These are projections of the attractor onto the xz-plane (left) and the xy-plane (right) for $r = 28$.