Review of Introductory Differential Equations

Note. We give a quick review of some of the important results from sophomore level Differential Equations (MATH 2120). Here, we concentrate on linear differential equations and existence of solutions results. For notes at this level, see my online notes based on Shepley Ross' *Introduction to Ordinary Differential Equations* 4th Edition at:

`http://faculty.etsu.edu/gardnerr/`

`Differential-Equations/DE-Ross4-notes.htm`.

Definition. An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a *differential equation* (or "DE"). A DE involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an *ordinary differential equation* (or "ODE"). A DE involving partial derivatives of one of more dependent variables with respect to more than one independent variable is called a *partial differential equation* (or "PDE"). The order of the highest ordered derivatives involved in a DE is called the *order* of the DE.

Definition. A *linear ODE* of order n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = G(x)$$

where $P_0(x)$ is not identically 0. The term $G(x)$ is called the *nonhomogeneous term.* If $G(x)$ is identically 0, then the DE is said to be *homogeneous.*

Definition. Consider the nth order ODE $F[x, y, y', \ldots, y^{(n)}] = 0$ wher F is a real function of its $n + 2$ arguments $x, y, y', \ldots, y^{(n)}$.

1. Let f be a real function defined for all x in an interval I and having an nth order derivative for all $x \in I$. Function f is called an *explicit solution* of the above DE on I if $F[, f, f', \ldots, f^{(n)}] = 0$ for all $x \in I$.

2. A relation $g(x, y) = 0$ is an *implicit solution* of the DE if it defines (implicitly) a function f of x on I that is an explicit solution of the DE.

Note. An nth order linear DE (under appropriate conditions) will have an *n-parameter family of solutions.*

Example. The second order ODE $y'' + y = 0$ has the 2-parameter family of solutions $y = c_1 \cos x + c_2 \sin x$.

Definition/Note. A DE can have *supplementary conditions* on the solution. If all the supplementary conditions relate to one independent value x then the problem of finding a solution to the DE satisfying the supplementary conditions is called an *initial value problem* (or "IVP"). If the supplementary conditions relate to two or more x values, the problem is called a *boundary value problem* (or "BVP").

Example. Solve the IVP:

$$\begin{aligned} y'' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

Example. Solve the BVP:

$$\begin{aligned} y'' + y &= 0 \\ y(0) &= 1 \\ y(\pi/2) &= 2. \end{aligned}$$

Note. We see in the previous two examples that the DE $y'' + y = 0$ has a 2-parameter family of solutions and the 2-parameters can be determined if 2 supplementary conditions are given. The 2 parameters are is some sense related to the following idea.

Definition. The n functions $f_1, f_2, \ldots, f_n$ are called *linearly dependent* on $a \leq x \leq b$ if there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $a \leq x \leq b$. n functions are called *linearly independent* on the interval $a \leq x \leq b$ if they are not linearly dependent there.

Note. We'll now state two existence and uniqueness theorems concerning linear ODEs.

Theorem R1. Basic Existence and Uniqueness Theorem for Linear Homogeneous ODEs.
The nth order homogeneous linear DE

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = 0$$

where $P_0, P_1, \ldots, P_n$ are continuous for $a \leq x \leq b$ and $P_0(x) \neq 0$ for $a \leq x \leq b$ has n linearly independent solutions. Also, if $f_1, f_2, \ldots, f_n$ are linearly independent solutions of the DE then any linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

is also a solution (that's the reason such DEs are called "linear"). In fact, any solution of the DE can be written as some linear combination of the f_i's.

Note. Recall from Introductory Differential Equations (MATH 2120) that we had the following methods to solve linear DEs:

1. homogeneous linear DEs with constant coefficients,

2. method of undetermined coefficients,

3. variation of parameters,

4. reduction of order, and

5. series solutions.

Theorem R2. Basic Existence and Uniqueness Theorem for Nonhomogeneous Linear ODEs.

Let y_p be a *particular solution* of the linear nonhomogeneous DE

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = G(x)$$

and let y_c be the general solution (complementary function) of the associated linear homogeneous DE

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = 0.$$

Then every solution of the original nonhomogeneous DE is of the form $y_c + y_p$.

Note. Theorem R2 is very reminiscent of the structure of solutions of a linear system of equations of the form $A\vec{x} = \vec{b}$. See my online notes for "Homogeneous Systems, Subspaces and Bases" for sophomore Linear Algebra (MATH 2010) at `http://faculty.etsu.edu/gardnerr/2010/c1s6.pdf`; in particular, see Theorem 1.18, "Structure of the Solution Set of $A\vec{x} = \vec{b}$.

Example. Solve $y'' + y = e^x$. Notice that $y_p = (1/2)e^x$.

Revised: 3/4/2019