DIFFERENTIAL EQUATIONS II, FINAL

11:15-12:30

NAME ______________________________ STUDENT NUMBER ______________________

Each problem is worth 20 points. Show all work. Be neat and use equal signs where applicable. Remember, you are not only trying to find the answer, but you are also trying to convince me that you know what you are doing! No calculators.

1. One of the advantages of solving systems of differential equations is that it allows one to also solve single linear differential equations using matrix methods. Consider the equation $ay'' + by' + cy = 0$. Transform this equation into a system of first order equations, $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Find the equation that determines the eigenvalues of $\mathbf{A}$. Note that this equation is the auxiliary (or characteristic) equation for the original differential equation.

2. Use the method of variation of parameters to find the general solution of

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

   Recall that the solution is given by $\mathbf{x} = \psi(t)\mathbf{u}(t)$ where $\psi(t)$ is a fundamental matrix of the system and $\mathbf{u}(t)$ satisfies $\psi(t)\mathbf{u}'(t) = \mathbf{g}(t)$ where $\mathbf{g}(t)$ is the nonhomogeneous part of the original differential equation.

3. Consider

$$\begin{cases} dx/dt &= x + y^2 \\ dy/dt &= x + y \end{cases}.$$

   Determine all real critical points and discuss their stability. Don't forget to show that the property of almost-linearity applies at the particular critical points.

4. Use mathematical induction to show that

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

   for $s > a$ where n is a positive integer.

5. Use the Laplace Transform to solve the initial value problem

   $y'' - 4y' + 4y = 0;$

   $y(0) = 1,$

   $y'(0) = 1.$