The Derivation of Kepler's Laws of Planetary Motion From Newton's Law of Gravity

Note. The German astronomer/astrologer Johannes Kepler (1571–1630) made the following observations about the movements of the planets around the Sun, expanding on a conjecture of the Polish astronomer Nicholas Copernicus (1473– 1543):

- 1. Each planet orbits the Sun in an elliptical orbit with the Sun at one focus.
- 2. A line drawn from the Sun to a planet sweeps out equal areas in equal times.
- 3. The square of the period of a planet is proportional to the cube of the mean distance between the planet and the Sun (if distance is measured in Astronomical Units, AUs, and time is measured in Earth years, then the constant of proportionality is 1).

Kepler's laws are purely empirical and, when stated, has no theoretical backing. It is interesting that Kepler reached all these conclusions from data gathered with the naked eye (primarily data that was collected by Tycho Brahe (1546–1601) since his investigations took place before the invention of the telescope by Galileo (1564– 1642) in 1610. Kepler's laws were simply observations, however. Therefore, one of the biggest questions of the Renaissance was: "Why (or perhaps more accurately, "how") do the planets obey Kepler's laws?" Sir Isaac Newton (1642–1727) answered this question with his Universal Law of Gravitation.





Johannes Kepler

Images from Wikipedia (2/13/2019).

Tycho Brahe



Galileo Galilei



Isaac Newton

Note. We will now take Newton's law of gravitation and derive Kepler's First Law. The following is in M. W. Hirsch and S. Smale's *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press, 1974); see Chapter 2. We start by considering a *vector field* $\vec{F}(\vec{x})$. this is a function which associates with each point (or vector) in 3-space (3 dimensional Euclidean space) another vector (or point) of 3-space. We want \vec{F} to describe the force due to gravity at the point \vec{x} .

Definition. Let $v(\vec{x})$ be a function that associates a scalar with each point \vec{x} of 3-space. Then the function $\vec{F}(\vec{x})$ is said to be *conservative* if

$$\vec{D}(\vec{x}) = -\left(\frac{\partial V}{\partial x_1}\hat{i} + \frac{\partial V}{\partial x_2}\hat{j} + \frac{\partial V}{\partial x_3}\hat{k}\right)$$

where we represent the coordinates of 3-space with x_1, x_2, x_3 and let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors pointing along the x_1, x_2, x_3 axes, respectively. The function $V(\vec{x})$ is called the *potential energy* function. **Note/Definition.** Recall that the *kinetic energy* of a particle is $\frac{1}{2} \times \max \times (\text{velocity})^2$. So if $\vec{x}(t)$ describes the position of a particle at time t, define its *kinetic energy* to be $T = \frac{1}{2}m||\vec{x}'||^2$ where m is the mass of the particle. The *total energy* of such a particle is

$$E = \text{potential energy} + \text{kinetic energy} = T + V.$$

Notice that T and V can both be viewed as functions of time. So, too can E:

$$E = E(t) = \frac{1}{2}m\|\vec{x}'\|^2 + V(\vec{x}(t)).$$

Theorem. Conservation of Energy Theorem.

Let $\vec{x}(t)$ be the trajectory of a particle moving in a conservative force field $\vec{F} = -\text{grad}(V)$. Then the total energy E is a constant.

Proof. We will show that $\frac{d}{dt}[E] = 0$. We have

$$\frac{d}{dt}[E] = \frac{d}{dt}[T+V] = \frac{d}{dt} \left[\frac{1}{2}m \|\vec{x}'\|^2 + V(\vec{x}(t)) \right].$$

Now, $\|\vec{x}'\|^2 = (x_1')^2 + (x_2')^2 + (x_3')^2$ where x_1, x_2, x_3 are each functions of time and $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_2\hat{k}$. So, with dot products represented by angled brackets (the conventional notation for an inner product),

$$\frac{d}{dt}[\|\vec{x}'\|^2] = 2x_1'x_1'' + 2x_1'x_2'' + 2x_3'x_3'' = 2\vec{x}'\cdot\vec{x}'' = 2\langle\vec{x}',\vec{x}''\rangle.$$

Also

$$\frac{d}{dt}[V(\vec{x}(t))] = \frac{\partial V}{\partial x_1} x_1' + \frac{\partial V}{\partial x_2} x_2' + \frac{\partial V}{\partial x_3} x_3' = \operatorname{grad}(V) \cdot \vec{x}' = \langle \operatorname{grad}(V), \vec{x}' \rangle.$$

So,

$$\frac{d}{dt}[E] = m\langle \vec{x}', \vec{x}'' \rangle + \langle \operatorname{grad}(V), \vec{x}' \rangle = m\langle \vec{x}'', \vec{x}' \rangle + \langle \operatorname{grad}(V), \vec{x}' \rangle$$
$$= \langle m\vec{x}'' + \operatorname{grad}(V), \vec{x}' \rangle = \langle m\vec{x}'' - \vec{F}(\vec{x}), \vec{x}' \rangle.$$

From Newton's Second Law, force = mass × acceleration, $m\vec{x}'' - \vec{F}(\vec{x}) = \vec{0}$ and so $\frac{d}{dt}[E] = \langle \vec{0}, \vec{x}' \rangle = 0$. Therefore, E is a constant and the total energy is conserved.

Note. The Conservation of Energy Theorem is why such force fields are called "conservative."

Definition. A force field $\vec{F}(\vec{x})$ is said to be *central* if $\vec{F}(\vec{x})$ is a scalar (function) multiple of \vec{x} , $\vec{F}(\vec{x}) = \lambda(x)\vec{x}$, where $\lambda(\vec{x})$ is a scalar.

Note. We now make two claims, but omit the proofs.

Lemma 1. Let \vec{F} be a conservative force field. If \vec{F} is central, then $\vec{F}(\vec{x}) = f(\|\vec{x}\|)\vec{x}$ for some scalar function $f(\|\vec{x}\|)$.

Lemma 2. A particle moving under the influence of a central field moves in a fixed plane.

Note. In light of Lemma 2, to derive Kepler's Laws, we need only consider motion in a plane (i.e., in 2-space). We now have

$$\vec{F}(\vec{x}) = -\left(\frac{\partial V}{\partial x_1}\hat{\imath} + \frac{\partial V}{\partial x_2}\hat{\jmath}\right) = -\text{grad}(V).$$

We will now convert to polar coordinates (r, θ) where $\|\vec{x}\| = r$.

Definition. For a particle moving in the (r, θ) -plane with mass m, define the angular momentum $h = mr^2 d\theta/dt$. (Notice that r and θ are functions of time, and therefore so is h.)

Theorem. Conservation of Angular Momentum.

For a particle moving in a central force field, angular momentum is constant.

Proof. Lets represent the (r, θ) -plane in terms of the complex plane. Then we can represent a given point (r, θ) as $re^{i\theta}$. In the (x_1, x_2) -plane,

$$x_1 = \operatorname{Re}(re^{i\theta})$$
 and $x_2 = \operatorname{Im}(re^{i\theta})$.

With $\vec{x} = r^{e^I \theta}$, we have

$$\vec{x}' = [r']e^{i\theta} + r[ie^{i\theta}]$$

$$\vec{x}'' = [r'']e^{i\theta} + r'[ie^{i\theta}\theta'] + [r'](ie^{i\theta}\theta') + r[[-e^{i\theta}\theta']\theta' + (ie^{i\theta})\theta'']$$

$$= e^{i\theta}(r'' - r(\theta')^2) + ie^{i\theta}(2r'\theta' + r\theta''). \qquad (*)$$

Now, by Lemma 1, $\vec{F}(\vec{x}) = f(r)\vec{x}$ and by Newton's Second Law, $\vec{f}(\vec{x}) = m\vec{x}''$. So

$$\vec{x}'' = \frac{1}{m} f(r) \vec{x} = \frac{1}{m} f(r) r e^{i\theta}.$$
 (**)

Equating (*) and (**) gives

$$\frac{r}{m}f(r) = (r'' + r(\theta')^2) + i(2r'\theta' + r\theta'').$$

Since the left-hand side is real, it must be that $2r'\theta^2 + r\theta'' = 0$ $2rr'\theta' + r^2\theta'' = 0$ or $\frac{d}{dt}[r^2\theta'] = 0$. So $r^2\theta' = r^2 d\theta/dt$ is constant and therefore angular momentum $h = mr^2 d\theta/dt$ is constant.

Theorem. Kepler's Second Law.

Assuming a conservative central force field emanating from the Sun, a line drawn from the Sun to a planet sweeps out equal areas in equal time.

Proof. Recall that in (r, θ) -plane the area bounded by $r(\theta)$ for $\theta \in [a, b]$ is

$$A = \int_{\theta=a}^{\theta=\beta} \frac{1}{2} (r(\theta))^2 \, d\theta$$

Treating $A, r, and \theta$ as functions of time:

$$\frac{DA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2m}h = \text{constant}.$$

So the rate of change of area with respect to time is a constant and Kepler's Second Law follows.

Note. Kepler's Second Law is simply a statement of Conservation of Angular Momentum (or course, that's not what *he* thought!). We now shift our attention to Kepler's First Law.

Note. Recall that Newton's Law of Gravitation says that if a mass m_1 lies at $\vec{0}$ and another mass m_2 lies at \vec{x} , then the force on m_2 is

$$-\frac{gm_1m_2}{\|\vec{x}\|^2}\frac{\vec{x}}{\|\vec{x}\|} = -\frac{gm_1m_2}{\|\vec{x}\|^3}\vec{x}$$

where g is the gravitational constant.

Note. we are going to assume m_1 is much greater than m_2 . So the acceleration of m_1 due to the force of gravity from m_2 will be negligible. So with m_1 unaccelerated, it remains at $\vec{0}$ and the force on m_2 is $\vec{F}(\vec{x}) = -C\vec{x}/||\vec{x}||^3$ for some constant C. (We need not make this assumption if we place the center of mass at $\vec{0}$.) Better yet, with a change of force units, we can say $\vec{F}(\vec{x}) = -\vec{x}/||\vec{x}||^3$. This force field is conservative since $\vec{F}(\vec{x}) = -\text{grad}(V)$ where $V(\vec{x}) = -1/||\vec{x}||$.

Problem 1. Prove V satisfies this relationship. HINT: Let $\vec{x}(t) = x_1(t)\hat{i} + x_2(t)\hat{j}$.

Note. Notice also that this force field is central and we need only consider motion in a plane. Again, we will stay in the (r, θ) -plane.

Note. Recall that $r(1 + \varepsilon \cos \theta) = \ell$ is a conic section of eccentricity ε .

Problem 2. Use the substitutions $r \cos \theta = x$ and $r^2 = x^2 + y^2$ to verify this statement. Show that for $0 < \varepsilon < 1$ the equation gives an ellipse, for $\varepsilon > 1$ a hyperbola, for $\varepsilon = 0$ a circle, and for $\varepsilon = 1$ a parabola.

Note. We introduce a new variable: u(t) = 1/r(t). Since θ is also a function of time, we can treat u as a function of θ .

Lemma 3. We have the following relationship:

kinetic energy
$$= T = \frac{1}{2} \frac{h^2}{m} \left(\left(\frac{du}{d\theta} \right)^2 + u^2 \right).$$

Proof. Recall that with $\vec{x} = re^{i\theta}$ as above, $\vec{x}' = r'e^{i\theta} + rie^{i\theta}\theta'$ and $\|\vec{x}'\|^2 = (r')^2 + (r\theta')^2$. So since $T = \frac{1}{2}m\|\vec{x}'\|^2$, we have

$$T = \frac{1}{2}m((r')^2 + (r\theta')^2)$$

Since r(t) = 1/u(t) or r = 1/u, we have

$$r' = -\frac{1}{u^2} \frac{du}{d\theta} \theta' \text{ by the Chain Rule}$$
$$= -r^2 \theta' \frac{du}{d\theta}$$
$$= -\frac{k}{m} \frac{du}{d\theta} \text{ since } h = mr^2 \theta'.$$

Also since $h = mr^2\theta'$, $r\theta' = h/(nr) = hu/m$. Therefore,

$$T = \frac{1}{2}m\left(\left(-\frac{h}{m}\frac{du}{d\theta}\right)^2 + \left(\frac{hu}{m}\right)^2\right) \text{ or } T = \frac{1}{2}\frac{h^2}{m}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right).$$

Theorem. Kepler's First Law.

Under Newton's Law of Gravitation (an "inverse square law") $\vec{F}(\vec{x}) = \vec{x}/||\vec{x}||^3$ planets (or anything else) orbits the sun in a conic section.

Proof. Since u = 1/r and $V = -1/\|\vec{x}\| = -1/r$, we have u = -V. Since T = E - V = E + u we have from Lemma 3,

$$T = \frac{1}{2}\frac{h^2}{m}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) = E + u \text{ or } \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{h^2}(E + u).$$

Differentiating with respect to θ :

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{2m}{h^2}\left(\frac{dE}{d\theta} + \frac{du}{d\theta}\right).$$

But E is constant with respect to time and with respect to θ so

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = \frac{2m}{h^2}\frac{du}{d\theta}.$$

Dividing by $2 du/d\theta$ gives

$$\frac{d^2u}{d\theta} + u = \frac{m}{h^2}.$$
(*)

It is this second order nonhomogeneous DE that will give the result. The general solution to (*) is

$$u = \frac{m}{h^2} + C\cos(\theta + \theta_0) \tag{**}$$

where C and θ_0 are arbitrary constants. We find that any solution of (*) can be put in the form

$$u = \frac{m}{h^2} \left(1 + \left(1 + 2\frac{Eh^2}{m} \right)^{1/2} \cos \theta \right)$$

or

$$r(1 + \varepsilon \cos \theta) = \frac{h^2}{m} = \ell \text{ where} \varepsilon = \left(1 + \frac{2Eh^2}{m}\right)^{1/2}.$$

Problem 3. Verify that (**) is the general solution to (*).

Note. Notice that if E < 0 then $\varepsilon < 1$ and the conic is an ellipse. If E = 0 then $\varepsilon = 0$ and the conic is a parabola. If E > 0 then $\varepsilon > 1$ and the conic is a hyperbola.

Bonus. In deriving (*), we assumed $du/d\theta \neq 0$. Why is this true, or what if $du/d\theta = 0$? ANSWER: Then dr/dt = 0 and the orbit is circular (so the Theorem still holds).

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