# DIFFERENTIAL EQUATIONS II, TEST II

## 11:15-12:30

NAME ______________________________ STUDENT NUMBER ________________

Each problem is worth 16 points. Show all work. Be neat and use equal signs where applicable. Remember, you are not only trying to find the answer, but you are also trying to convince me that you know what you are doing! No calculators.

1. Consider

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}.$$

   Find the eigenvalues of the coefficient matrix and use them to classify the critical point $(0,0)$ as stable, asymptotically stable, or unstable.

2. Find an equation for the trajectories of

$$\frac{d^2\theta}{dt^2} + \theta - \alpha\theta^3 = 0,$$

   where $\alpha$ is a constant. That is, let $x = \theta$ and $y = d\theta/dt$ and find a relationship between $x$ and $y$.

3. Consider

$$\begin{cases} dx/dt &= y \\ dy/dt &= -x + \mu y(1 - x^2), \mu > 0 \end{cases}.$$

   Show that the system is almost linear near the critical point $(0,0)$ and discuss the stability of the original system at $(0,0)$. Is $(0,0)$ a node, center, saddle point, or spiral point of the associated linear system?

4. The system

$$\begin{cases} dx/dt &= x(1 - 0.5y) \\ dy/dt &= y(-0.75 + 0.25x) \end{cases}$$

   is a predator-prey system of equations. $(0,0)$ is a saddle point for the system. Find another critical point of the nonlinear system and classify this critical point for the associated linear system. Sketch some trajectories for the original system.

5. Consider

$$\begin{cases} dx/dt &= -x - y - x^3 \\ dy/dt &= x - y - y^3 \end{cases}.$$

Construct a Liapunov function of the form $Ax^2 + By^2$ and determine whether $(0,0)$ is asymptotically stable, stable, or unstable.

6. Consider

$$\begin{cases} dx/dt &= 4x - 4y - x(x^2 + y^2) \\ dy/dt &= 4x + 4y - y(x^2 + y^2) \end{cases}.$$

Transform this system into polar coordinates.

HINT: $r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$ and $r^2\frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt}$.

Apply the Poincaré-Bendixson Theorem to show that there exists a closed trajectory somewhere between the circles $x^2 + y^2 = \frac{1}{4}$ and $x^2 + y^2 = 16$.