Section 4.3. The Method of Undetermined Coefficients

Note. In this section we deal with the nonhomogeneous linear DE with constant coefficients:

\[ a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x) \]

where \( a_0, a_1, \ldots, a_n \) are real constants and \( F(x) \) is a function of \( x \).

Example. Find the general solution of \( y'' - 2y' - 3y = 2e^{4x} \).

Solution. The associated homogeneous DE is \( y'' - 2y' - 3y = 0 \). The general solution of this homogeneous DE is \( y_c = c_1 e^{-x} + c_2 e^{3x} \). Recall that this is the complementary function of the original DE. For a particular solution of the original nonhomogeneous DE, let's try \( y = Ae^{4x} \). Substituting in to the original DE we find that \( A = \frac{2}{5} \) works so that \( y_p = \frac{2}{5} e^{4x} \) is a particular integral. So the general solution of the DE is

\[ y = \frac{2}{5} e^{4x} + c_1 e^{-x} + c_2 e^{3x}. \]

Example. Find the general solution of the DE: \( y'' - 2y' - 3y = 2e^{3x} \).

Solution. The complementary function is (as above) \( y = c_1 e^{-x} + c_2 e^{3x} \). If we try for a particular solution of the nonhomogeneous DE of the form \( y = Ae^{3x} \). It doesn’t work!!! We need more knowledge.
Definition. A function is a *UC function* if it is either one of the following:

1. $x^n$ where $n$ is a nonnegative integer,
2. $e^{ax}$ where $a \neq 0$,
3. $\sin(bx + c)$ where $b \neq 0$,
4. $\cos(bx + c)$ where $b \neq 0$,

or a finite product of two or more functions of these four types.

Definition. Consider a UC function $f$. The set of functions consisting of $f$ itself and all linearly independent UC functions, of which the successive derivatives of $f$ are either constant multiples or linear combinations, will be called the *UC set of $f$* (or the *differential family of $f$*).

Example. Page 151 Example 4.33. Consider $f(x) = x^2 \sin x$. Function $f$ is a product of two UC functions and so is itself a UC function. We now find the UC set of $f$. We have

$$f'(x) = 2x \sin x + x^2 \cos x$$
$$f''(x) = 2 \sin x + 4x \cos x - x^2 \sin x$$
$$f'''(x) = 6 \cos x - 6x \sin x - x^2 \cos x$$

and we see that the UC set of $f$ is $S = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$. Notice that the UC set of $f_1(x) = x^2$ is $S_1 = \{x^2, x, 1\}$ and the UC set of $f_2(x) =$
sin \( x \) is \( S_2 = \{ \sin x, \cos x \} \). In fact, any element of \( S \) is a product of an element of \( S_1 \) and an element of \( S_2 \). This foreshadows the following.

**Theorem.** Suppose \( h \) is a UC function defined as the product \( fg \) of two basic UC functions \( f \) and \( g \) (that is, \( f \) and \( g \) are of the four types described above). Then the UC set of \( h \) is the set of all products obtained by multiplying the various members of the UC set of \( f \) by the various members of the UC set of \( g \).

**Example.** We can easily find the UC set of \( f(x) = x^3e^x \). The UC set for \( x^3 \) is \( S_1 = \{ x^3, x^2, x, 1 \} \) and the UC set for \( e^x \) is \( S_2 = \{ e^x \} \). So by the previous theorem, the UC set of \( f \) is \( S = \{ x^3e^x, x^2e^x, xe^x, e^x \} \).

**Note.** We now give an outline of the *Method of Undetermined Coefficients* and then illustrate it with several examples.

**Note.** The Method of Undetermined Coefficients is as follows. Consider the DE

\[
a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = F(x)
\]

where \( F(x) \) is a finite linear combination \( F = A_1u_1 + A_2u_2 + \cdots + A_mu_m \) of UC functions \( u_1, u_2, \ldots, u_m \). Suppose that \( y_c \) is the complementary solution of the DE.

1. For each UC function \( u_1, u_2, \ldots, u_m \), form the corresponding UC sets \( S_1, S_2, \ldots, S_m \).

2. Suppose that \( S_j \subset S_k \). Then omit \( S_j \) from the collection of \( S \)'s.
3. Suppose that \( S_\ell \) includes one or more members which is a solution to the corresponding homogeneous DE. Then multiply each member of \( S_\ell \) by the lowest (positive integer) power of \( x \) which produces in \( S_\ell \) a new collection of functions none of which are a solution to the associated homogeneous DE.

4. Form a linear combination with all of the members of the \( S \)’s. We now need to determine the coefficients of this linear combination.

5. Determine the coefficients by substituting into the original DE and equating similar terms.

Example. Let’s now return to \( y'' - 2y' - 3y = 2e^{3x} \). The nonhomogeneous term \( 2e^{3x} \) is a linear combination of the UC function \( e^{3x} \). So let \( S = \{e^{3x}\} \). However, by Step 3, \( e^{3x} \) is a solution to the associated homogeneous DE. So by Step 3, replace \( e^{3x} \) with \( xe^{3x} \). And so \( S = \{xe^{3x}\} \). Notice this is not a solution of the associated homogeneous DE. So let \( y_p = Axe^{3x} \). The \( y' = Ae^{3x} + 3Axe^{3x} \) and \( y'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} = 6Ae^{3x} + 9Axe^{3x} \), so we substitute into the nonhomogeneous DE to get \( (6Ae^{3x} + 9Axe^{3x}) - 2(Ae^{3x} + 3Axe^{3x}) - 3(Axe^{3x}) = 4Ae^{3x} = 2e^{3x} \) so that \( A = 1/2 \) and \( y_p = \frac{1}{2}e^{3x} \). So the general solution to the nonhomogeneous DE is

\[
y = \frac{1}{2}e^{3x} + c_1e^{-x} + c_2e^{3x}.
\]

Example. Page 160 Number 11. Find the general solution of \( y'' + 4y = 4 \sin(2x) + 8 \cos(2x) \).
Solution. The associated homogeneous DE is \( y'' + 4y = 0 \) and has general solution \( y_c = c_1 \sin(2x) + c_2 \sin(2x) \). The nonhomogeneous term is \( F(x) = 4 \sin(2x) + 8 \cos(2x) \). So \( F(x) \) is a linear combination of the UC functions \( \sin(2x) \) and \( \cos(2x) \). Each has the UC set \( S = \{ \sin(2x), \cos(2x) \} \). From Step 3, \( \sin(2x) \) and \( \cos(2x) \) are both solutions of the associated DE so replace these with \( S = \{ x \sin(2x), x \cos(2x) \} \). From Step 4, suppose \( y_p = Ax \sin(2x) + Bx \cos(2x) \). Substituting into the nonhomogeneous DE we find that \( A = 2 \) and \( B = -1 \), so \( y_p = 2x \sin(2x) - x \cos(2x) \). Then the general solution is

\[
y = y_p + y_c = 2x \sin(2x) - x \cos(2x) + c_1 \sin(2x) + c_2 \cos(2x).
\]