Section 5.6. Electric Circuit Problems

Note. In this section, we consider certain second order linear DEs which result from an elementary circuit and Kirchoff’s Voltage Law.

Note. Recall:

1. *Electromotive force* (emf) is the difference in potential of the two terminals of an energy source (commonly a battery, or cell). We measure emf (or voltage) in volts (V).

2. *Current* is the charge that passes through a cross section of wire in a unit time. Current is measured in amperes and charge is measured in coulombs.

3. The *resistance* to the flow of current is measured in ohms (Ω).

4. An *inductor* coil prevents a battery from instantaneously establishing a current through it. The battery (emf) has to do work against the coil as the current flows through it. Inductance is measured in henrys (H).

5. A *capacitor* consists of two pieces of metal (usually plates) that store charge. Capacitance is measured in farads (or coulombs per volt).

Definition/Note. The voltage drops across a resistor is \( E_R = Ri \) where \( R \) is the resistance and \( i \) is current. The voltage drop across an inductor is \( E_L = Li' \) where \( L \) is the inductance (and the prime indicates a derivative with respect to time). The voltage drop across a capacitor is \( E_C = \frac{1}{C}q \) where \( C \) is the capacitance and \( q \) is the instantaneous charge on the capacitor. From the definitions above, \( q' = i \) and so \( E_C = \frac{1}{C} \int i \, dt \).
**Note.** *Kirchoff’s Voltage Law* relates these voltage drops: The sum of the voltage drops across resistors, inductors, and capacitors is equal to the total emf in a closed circuit.

![Figure 5.12](image)

This circuit, has the equation (from Kirchoff’s Law)

$$Li' + Ri + \frac{1}{C}q = E.$$  

Both $i$ and $q$ are functions of time, so we can write (since $i = q'$):

$$Lq'' + Rq' + \frac{1}{C}q = E$$

where $q$ is charge at time $t$.

**Note.** We can also write a DE in terms of current $i$: $Li' + Ri + \frac{1}{C}q = E$ or

$$Li'' + Ri' + \frac{1}{C}q' = E',$$

or

$$Li'' + Ri' + \frac{1}{C}i = E'.$$

Notice that we have two second order linear DEs, one involving charge $q$ and one involving current $i$. If the current involves no capacitor, then we get the DE governing the so called $LR$-circuit:

$$Li' + RiE.$$
Also, if there is no inductor, we have

\[ Rq' + \frac{1}{C}q = E. \]

**Example.** Page 232 Number 5. A circuit has in series of electromotive force given by \( E(t) = 100 \sin 200t \) V, a resistor of 40 \( \Omega \), an inductor of 0.25 H, and a capacitor of \( 4 \times 10^{-4} \) farads. If the initial current is zero, and the initial charge on the capacitor is 0.01 coulombs, find the current at any time \( t > 0 \).

**Solution.** Symbolically, we have \( E(t) = 100 \sin 200t, \ R = 40, \ L = 0.25, \ C = 4 \times 10^{-4}, \ q(0) = 0.01, \) and \( i(0) = q'(0) = 0 \). So using

\[ Lq'' + Rq' + \frac{1}{C}q = E \]

we get \( 0.25q'' + 40q' + 2500a = 100 \sin 200t \) or

\[ q'' + 160q' + 10,000q = 400 \sin 200t. \]

The associated homogeneous DE is \( q'' + 160q' + 10,000q = 0 \) which has auxiliary equation \( m^2 + 160m + 10,000 = 0 \) so that we have

\[ m = \frac{-160 \pm \sqrt{(160)^2 - 40,000}}{2} = -80 \pm 60i. \]

So

\[ q_c = e^{-80t} (c_1 \cos 60t + c_2 \sin 60t). \]

By the method of undetermined coefficients, we look for:

\[ q_p = A \sin 200t + B \cos 200t \]
\[ q'_p = 200A \cos 200t - 200B \sin 200t \]
\[ q''_p = -40,000A \sin 200t - 40,000B \cos 200t. \]
So we need

\[-40,000(A \sin 200t + B \cos 200t) + 32,000(A \cos 200t - B \sin 200t)\]

\[+ 10,000(A \sin 200t + B \cos 200t) = 400 \sin 200t\]

or

\[-62,000A \sin 200t + 2,000B \cos 200t = 400 \sin 200t.\]

So we must have \(A = -1/155\) and \(B = 0\) and \(q_p = -(1/155) \sin 200t\). So we have

\[q = e^{-80t}(c_1 \cos 60t + c_2 \sin 60t) - \frac{1}{155} \sin 200t.\]

Since \(q(0) = 0.01\) then \(c_1 = 0.01\) and so

\[q = e^{-80t}(0.01 \cos 60t + c_2 \sin 60t) - \frac{1}{155} \sin 200t.\]

Then

\[q' = -80e^{-80t}(0.01 \cos 60t + c_2 \sin 60t) + e^{-80t}(-60(0.01) \sin 60t + 60c_2 \cos 60t) - \frac{200}{155} \cos 200t\]

\[= e^{-80t}(-0.8 \cos 60t - 80c_2 \sin 60t - 0.6 \sin 60t + 60c_2 \cos 60t) - \frac{40}{31} \cos 200t\]

\[= e^{-80t}((-0.8 + 60c_2) \cos 60t + (-80c_2 - 0.6) \sin 60t) - \frac{40}{31} \cos 200t.\]

So \(q'(0) = -0.8 + 60c_2 - 40/31 = 0\) or \(60c_2 = 8/10 + 40/31 = 648/310 = 324/155\) and \(c_2 = 324/9300 = 27/775.\) So

\[q = e^{-80t}\left(0.1 \cos 60t + \frac{27}{775} \sin 60t\right) - \frac{1}{155} \sin 200t\]

and, since \(-0.8 + 60c_2 = 40/31\) and \(-80c_2 - 0.6 = -105/31,\)

\[i = q' = e^{-80t}((-0.8 + 60c_2) \cos 60t + (-80c_2 - 0.6) \sin 60t) - \frac{40}{31} \cos 200t.\]
\[ i = e^{-80t} \left( \frac{40}{31} \cos 60t - \frac{105}{31} \sin 60t \right) - \frac{40}{31} \cos 200t. \]

Notice that as time increases, the first term in \( i \) rapidly approaches zero (this is called the \textit{transient term}) and the second term (the \textit{steady state term}) dominates.