Section 6.2. Solutions About Singular Points;  
The Method of Frobenius

Note. We again consider the DE

\[ a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \]

This time we wish to find a solution defined “near” a singular point \( x_0 \) where \( a_0(x_0) = 0 \). For this, we give a classification of singular points.

Definition. With the notation established, let \( x_0 \) be a singular point of the above DE. If the functions

\[
\begin{align*}
\lim_{x \to x_0} (x - x_0) P_1(x) & \quad \text{if } x = x_0 \\
(x - x_0) P_1(x) & \quad \text{if } x \neq x_0
\end{align*}
\]

and

\[
\begin{align*}
\lim_{x \to x_0} (x - x_0)^2 P_2(x) & \quad \text{if } x = x_0 \\
(x - x_0)^2 P_2(x) & \quad \text{if } x \neq x_0
\end{align*}
\]

are both analytic at \( x_0 \), then \( x_0 \) is a regular singular point of the DE. If either of these new functions is not analytic at \( x_0 \), then \( x_0 \) is an irregular singular point of the DE.

Note. We follow the notation of Ross and denote the “new” functions as \((x - x_0)P_1(x)\) and \((x - x_0)P_2(x)\), even though they are defined as

\[
(x - x_0)P_1(x) = \begin{cases} 
\lim_{x \to x_0} (x - x_0)P_1(x) & \text{if } x = x_0 \\
(x - x_0)P_1(x) & \text{if } x \neq x_0
\end{cases}
\]
and
\[(x - x_0)^2 P_2(x) = \begin{cases} 
\lim_{x \to x_0} (x - x_0)^2 P_2(x) & \text{if } x = x_0 \\
(x - x_0)^2 P_2(x) & \text{if } x \neq x_0
\end{cases}\]

**Note.** We can find series solutions about such singular points.

**Theorem 6.2.** Suppose \(x_0\) is a regular singular point of
\[a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.\]
Then the DE has at least one nontrivial solution of the form
\[|x - x_0|^R \sum_{n=0}^{\infty} c_n (x - x_0)^n\]
where \(R\) is a (real or complex) constant which may be determined. This solution is valid in some deleted interval \(0 < |x - x_0| < s\) where \(s > 0\).

**Note.** We use the Method of Frobenius when we apply Theorem 6.2.

**Example.** Page 254 Number 11. Find solutions of
\[2x^2 y'' - xy' + (x - 5)y = 0\]
in some deleted interval \(0 < x < R\).

**Solution.** Notice that \(P_1(x) = -x/(2x^2)\) and \(P_2(x) = (x - 5)/(2x^2)\). Now with \(x_0 = 0\), we have
\[x P_1(x) = \frac{-x^2}{2x^2} = \frac{-1}{2}.\]
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\[ x^2 P_2(x) = \frac{x^2(x - 5)}{2x^2} = \frac{x - 5}{2} \]

where we use the red = as described above. So both of these new functions are analytic at \( x = 0 \). That is, \( x = 0 \) is a regular singular point. So assume

\[ y = |x|^R \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+R} \]

(ignoring the absolute value for now) is a solution. Then

\[ y' = \sum_{n=0}^{\infty} (n + R)c_n x^{n+R-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n + R)(n + R - 1)c_n x^{n+R-2} \]

Plugging these series into the DE gives

\[ 2 \sum_{n=0}^{\infty} (n + R)(n + R - 1)c_n x^{n+R} - \sum_{n=0}^{\infty} (n + R)c_n x^{n+R} + \sum_{n=0}^{\infty} c_n x^{n+R-1} - 5 \sum_{n=0}^{\infty} c_n x^{n+R} = 0. \]

Simplifying we get:

\[ (2R(R-1) - R - 5)c_0 x^R + \sum_{n=1}^{\infty} ((2(n + R)(n + R - 1) - (n + R) - 5)c_n + c_{n-1}) x^{n+R} = 0. \]

This means that

\[ 2R(R - 1) - R - 5 = 2R^2 - 3R - 5 = (2R - 5)(R + 1) = 0 \text{ or } R = -1, 5/2. \]

This is called the indicial equation. So we get the recurrence formula

\[ c_n = \frac{-c_{n-1}}{2(n + R)(n + R - 1) - (n + R) - 5} \quad \text{for } n \geq 1. \]

With \( R = 5/2 \), the recurrence formula becomes:

\[ c_n = \frac{-c_{n-1}}{n(2n + 7)} \quad \text{for } n \geq 1. \]

Notice that \( c_0 \) is arbitrary, and we get

\[ y = c_0 \left( x^{5/2} - \frac{1}{9} x^{7/2} + \frac{1}{198} x^{9/2} - \frac{1}{7722} x^{11/2} + \cdots \right). \]
If we use $R = -1$, we get

$$y = c_0 \left( x^{-1} + \frac{1}{5} + \frac{1}{30}x + \frac{1}{90}x^2 + \cdots \right).$$

Again $c_0$ is arbitrary. So the general solution is:

$$y = k_1 \left( x^{5/2} - \frac{1}{9} x^{7/2} + \frac{1}{198} x^{9/2} - \frac{1}{7722} x^{11/2} + \cdots \right) + k_2 \left( x^{-1} + \frac{1}{5} x + \frac{1}{30} x^2 + \frac{1}{90} x^3 + \cdots \right)$$

$$= k_1 x^{5/2} \left( 1 - \frac{1}{9} x + \frac{1}{198} x^2 - \frac{1}{7722} x^3 + \cdots \right) + k_2 x^{-1} \left( 1 + \frac{1}{5} x + \frac{1}{30} x^2 + \frac{1}{90} x^3 + \cdots \right).$$

Note. Notice that Theorem 6.2 guarantees at least one solution of a certain form. This previous example had two solutions of that form. The following theorem clarifies this a bit.

**Theorem 6.3.** Let $x_0$ be a regular singular point of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Let $R_1$ and $R_2$ be the roots of the indicial equation (where $R_1 > R_2$ for $R_1$ and $R_2$ real). Then

1. If $R_1 - R_2 \not\in \{0, 1, 2, \ldots\}$ then the DE has two nontrivial independent solutions $y_1$ and $y_2$ of the forms:

$$y_1(x) = |x - x_0|^R_1 \sum_{n=0}^\infty c_n(x - x_0) \quad \text{and} \quad y_2(x) = |x - x_0|^R_2 \sum_{n=0}^\infty c_n^*(x - x_0)^n$$

where $c_0 \neq 0$ and $c_0^* \neq 0$. 
2. If $R_1 - R_2 \in \{1, 2, 3, \ldots \}$ then the DE has two nontrivial linearly independent solutions $y_1$ and $y_2$ of the forms:

$$y_1(x) = |x - x_0|^{R_1} \sum_{n=0}^{\infty} c_n (x - x_0)$$ and

$$y_2(x) = |x - x_0|^{R_2} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n + cy_1(x) \ln |x - x_0|$$

where $c_0 \neq 0$, $c_0^* = 0$, and $c$ is a constant (possibly 0).

3. If $R_1 = R_2$ then the DE has two nontrivial linearly independent solutions $y_1$ and $y_2$ of the forms

$$y_1(x) = |x - x_0|^{R_1} \sum_{n=0}^{\infty} c_n (x - x_0)$$ and

$$y_2(x) = |x - x_0|^{R_1+1} \sum_{n=0}^{\infty} c_n^* (x - x_0)^n + y_1(x) \ln |x - x_0|$$

where $c_0 \neq 0$.

In each of these cases, the solutions are valid for $0 < |x - x_0| < R$ for some $R > 0$.

Example. Page 270 Number 16. Find solutions of

$$x^2 y'' + xy' + (x^2 - 1/4)y = 0$$

in some deleted interval $0 < x < R$.

Solution. Notice that $x_0 = 0$ is a regular singular point. So suppose $y = \sum_{n=0}^{\infty} c_n x^{n+R}$. Then calculating derivatives and plugging them into the DE gives

$$\left( R(R-1) + R - \frac{1}{4} \right) c_0 x^R + (1 + R)^2 - \frac{1}{4} c_1 x^{1+R}$$
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\[ + \sum_{n=2}^{\infty} \left( (n + R)(n + R - 1) + (n + R) - \frac{1}{4} \right) c_n + c_{n-2} \right) x^{n+R} = 0. \]

So from the indicial equation:

\[ R(R - 1) + R - \frac{1}{4} = 0 \text{ and } R_1 = \frac{1}{2}, R_2 = -\frac{1}{2}. \]

Notice that \( R_1 - R_2 = 1 \) and Case 2 of Theorem 6.3 applies. Using \( R = 1/2 \), we find that (as in the previous example):

\[ y_1(x) = c_1 x^{1/2} \left( 1 - \frac{x^2}{6} + \frac{x^5}{120} - \cdots \right). \]

Now, we hope that for \( R = -1/2 \), we get \( y_2(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \). If this leads to a contradiction, we will have to use reduction of order using \( y_1(x) \). With \( R = -1/2 \) we have \( c_0 = c_0, c_1 = c_1 \) (that is, \( c_0 \) and \( c_1 \) are arbitrary), and

\[ c_n = \frac{-c_{n-2}}{n^2 - n} \text{ for } n \geq 2. \]

We get

\[ y_2(x) = c_0 x^{-1/2} \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \right) + c_1 x^{1/2} \left( 1 - \frac{x^2}{6} + \frac{x^5}{120} - \cdots \right). \]

Notice that the second part of \( y_2(x) \) is \( y_1(x) \). In fact, \( y_2(x) \) is the general solution of the given DE.

**Note.** The previous example illustrates the fact that when \( R_1 - R_2 \) is a positive integer, it may be the case that the smaller root \( R_2 \) may generate the general solution. Therefore, it is a good habit to always use the smaller root first.