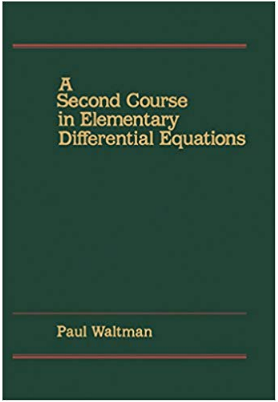
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.2. Some Elementary Matrix Algebra—Proofs of Theorems



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Theorem 1.2.1(1)

Theorem 2.1. Let $\alpha \in \mathbb{R}$ and suppose the products below are defined. Then

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Proof. Let A be $m \times p$, B be $p \times n$, and C be $n \times r$. Let $D = A(BC)$ and $E = (AB)C$. Then

$$d_{ij} \sum_{k=1}^{p} a_{ik} \left(\underbrace{\sum_{\ell=1}^{n} b_{k\ell} c_{\ell j}}_{(bc)_{kj}} \right) = \sum_{\ell=1}^{n} \left(\sum_{k=1}^{p} a_{ik} b_{k\ell} c_{\ell j} \right) = \sum_{\ell=1}^{n} \left(\underbrace{\sum_{k=1}^{p} a_{ik} b_{k\ell}}_{(ab)_{i\ell}} \right) c_{\ell j} = e_{ij}.$$

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Proof. If $AA^{-1} = \mathcal{I}$ exists then by Theorem 1.2.3,

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Proof. Suppose the columns of A are $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ and let $c_1, c_2, \ldots, c_n$ be scalars such that $c_1\vec{x}_1 + \vec{c}_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$. This is equivalent to $A\vec{v} = \vec{0}$ where $\vec{c} = [c_i]$.

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A is nonsingular if and only if $A\vec{c} = \vec{0}$ has a unique solution by Theorem 1.2.5. So if A is nonsingular, then $\vec{c} = \vec{0}$ and the columns of A are linear independent. If A is singular, then there is some $\vec{c} \neq \vec{0}$ satisfying $A\vec{c} = \vec{0}$ and the columns of A are linearly dependent. □

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