Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.3. The Structure of Solutions of Homogeneous Linear Systems—Proofs of Theorems

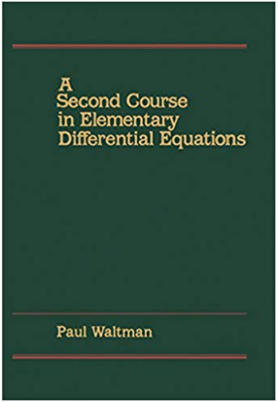

Table of contents

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Proof. Let $\vec{x}_1(t)$ and $\vec{x}_2(t)$ be differentiable vector functions and let $c_1, c_2 \in \mathbb{R}$. Then

$$L[c_1\vec{x}_1 + c_2\vec{x}_2] = (c_1\vec{x}_1 + c_2\vec{x}_2)' - A(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\vec{x}_1' + c_2\vec{x}_2' - c_1A\vec{x}_1 - c_2A\vec{x}_2$$

$$= c_1(\vec{x}_1' - A\vec{x}_1) + c_2(\vec{x}_2' - A\vec{x}_2) = c_1L[\vec{x}_1] + c_2L[\vec{x}_1].$$

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Theorem 1.3.3

Theorem 1.3.3. Let L and A be as in Theorem 1.3.2. If $\vec{x}_1$ and $\vec{x}_2$ are solutions of $\vec{x}\,' = A\vec{x}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution.

Proof. Define $L[\vec{x}] = \vec{x}\,' - A\vec{x}$. By Theorem 1.3.2, L is a linear operator. Since $\vec{x}_1$ and $\vec{x}_2$ are solutions of $\vec{x}\,' = A\vec{x}$, then $L[x_1] = L[\vec{x}_2] = \vec{0}$. So for $c_1, c_2 \in \mathbb{R}$, we have

$$L[c_1\vec{x}_1 + c_2\vec{x}_2] = c_1 L[\vec{x}_1] + c_2 L[\vec{x}_2] = \vec{0} + \vec{0} = \vec{0},$$

so that $L[c_1\vec{x}_1 + c_2\vec{x}_2] = \vec{0}$, or $(c_1\vec{x}_1 + c_2\vec{x}_2)' = A(c_1\vec{x}_1 + c_2\vec{x}_2)$. So a linear combination of $\vec{x}_1$ and $\vec{x}_2$ is a solution to $\vec{x}\,' = A\vec{x}$, as claimed. □

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Theorem 1.3.4

Theorem 1.3.4. Let L be as in Theorem 1.3.2. If Φ is the fundamental matrix for $L[\vec{x}] = \vec{0}$ on an interval I where $A(t)$ is continuous, then every solution of $L[\vec{x}] = \vec{0}$ can be written as $\phi\vec{c}$ for some constant vector $\vec{c}$.

Proof. Let $\vec{x}(t)$ be a solution on I and let $t_0 \in I$. Since the columns of Φ are linearly independent, $\Phi^{-1}(t_0)$ exists (by Theorem 1.2.4). Let $\vec{c} = \Phi^{-1}(t_0)\vec{x}(t_0)$. Then as seen above, $\vec{y}(t) = \Phi(t)\vec{c}$ is a solution to $L[\vec{x}] = \vec{0}$.

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Since this holds for all $t_0 \in I$, then $\vec{x}(t) = \vec{y}(t) = \Phi(t)\vec{c}$. □

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