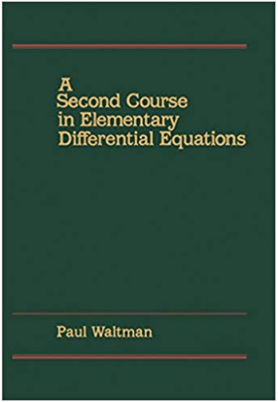
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.4. Matrix Analysis and Matrix Exponentiation—Proofs of Theorems



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Theorem 1.4.A

Theorem 1.4.A. Let A be an $R \times R$ matrix. Then:

5. $\|A\vec{x}\| \le \|A\|\|\vec{x}\|$ for $\vec{x}$ an R-vector.

Proof. (This is Exercise 1.4.2.) We have

$$\|A\vec{x}\| = \|[b_j]^T\| = \left\| \left[\sum_{i=1}^{R} a_{ji}x_i \right] \right\| = \sum_{j=1}^{R} \left| \sum_{i=1}^{R} a_{ji}x_i \right|$$

$$\le \sum_{j=1}^{R}\sum_{i=1}^{R} |a_{ji}x_i| = \sum_{i=1}^{R} |x_i| \sum_{j=1}^{R} |a_{ji}| \le \sum_{i=1}^{R}\sum_{i'=1}^{R} |x_{i'}| \sum_{j=1}^{R} |a_{ji}|$$

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Theorem 1.4.1

Theorem 1.4.1. Every Cauchy sequence of matrices (with real entries) A_n has a limit.

Proof. Let a_{ij}^p be the (i, j)th entry of matrix A_p. Then $|a_{ij}^n - a_{ij}^m| \leq \|A_n - A_m\|$ for all n, m. So the sequence of (i, j) entries form a Cauchy sequence of real numbers and therefore converges to say a_{ij}. Let $A = [a_{ij}]$. For $\varepsilon > 0$, choose N_{ij} such that $|a_{ij} - a_{ij}^n| < \varepsilon/R^2$ for all $n > N_{ij}$.

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$$\|A - A_n\| = \sum_{i,j} |a_{ij} - a_{ij}^n| < R^2(\varepsilon/R^2) = \varepsilon$$

for $n \geq N$. Therefore $A_n \to A$. □

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Theorem 1.4.2

Theorem 1.4.2. The series $I + \sum_{n=1}^{\infty} A^n.n!$ converges for all square matrices A.

Proof. Consider the partial sums $S_n = \sum_{k=0}^{n} A^k/k!$. We have

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} \frac{A^k}{k!} \right\| \leq \sum_{k=m+1}^{n} \frac{\|A^k\|}{k!} \leq \sum_{k=m+1}^{n} \frac{\|A\|^k}{k!} \leq \sum_{k=m+1}^{\infty} \frac{\|A\|^k}{k!}.$$

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Now $\sum_{k=0}^{\infty} \|A\|^k/k! = e^{\|A\|}$, so m can be chosen sufficiently large so that $\sum_{k=m+1}^{n} \|A\|^k/k! < \varepsilon$ for any given ε. So, S_n is Cauchy and so $\sum_{n=0}^{\infty} A^n/n!$ converges. □

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Theorem 1.4.4. For any square matrix M, $\det(e^M) \neq 0$.

Proof. (This is Exercise 1.4.10.) By Exercise 1.4.7, if $AB = BA$ then $e^A e^B = e^{A+B}$. Since M and $-M$ commute under multiplication (by Exercise 1.4.9), $e^M e^{-M} = e^0 = \mathcal{I}$. So e^M has as its inverse e^{-M} and e^M is invertible. So $\det(e^M) \neq 0$, as claimed. □

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