Chapter 1. Systems of Linear Differential Equations
Section 1.5. The Constant Coefficient Case: Real and Distinct Eigenvalues
Proofs of Theorems

Theorem 1.5.1

**Theorem 1.5.1.** Let $A$ be a constant matrix. A fundamental matrix $\Phi$ for $y' = Ay$ is $\Phi = e^{At}$.

**Proof.** We have $\frac{d}{dt}[e^{At}] = Ae^{At}$ and $\det(e^{At}) \neq 0$ by Theorem 1.4.4.

Theorem 1.5.2

**Theorem 1.5.2.** If $B = TAT^{-1}$, then $A$ and $B$ have the same eigenvalues.

**Proof.** First, $B - \lambda I = TAT^{-1} - \lambda TT^{-1} = T(A - \lambda I)T^{-1}$. By Theorem 1.2.3,

$$\det(B - \lambda I) = \det(T)\det(A - \lambda I)\det(T^{-1}).$$

Since $T$ (and $T^{-1}$) are invertible, then $\det(T) \neq 0$ and $\det(T^{-1}) \neq 0$ by Theorem 1.2.4. So $\det(B - \lambda I) = 0$ if and only if $\det(A - \lambda I) = 0$, so that $A$ and $B$ have the same eigenvalues, as claimed.

Theorem 1.5.3

**Theorem 1.5.3.** If $A$ is a constant matrix, $\lambda$ is an eigenvalue of $A$ and $\vec{c}$ a corresponding eigenvector, then $\vec{y} = e^{\lambda t}\vec{c}$ is a solution of $\vec{y}' = A\vec{y}$.

**Proof.** First,

$$A\vec{y} = Ae^{\lambda t}\vec{c} = e^{\lambda t}(A\vec{c}) = e^{\lambda t}(\lambda\vec{c}) = \lambda e^{\lambda t}\vec{c}.$$

Secondly,

$$\vec{y}' = \frac{d}{dt}[e^{\lambda t}\vec{c}] = \lambda e^{\lambda t}\vec{c}.$$

So the claim holds.