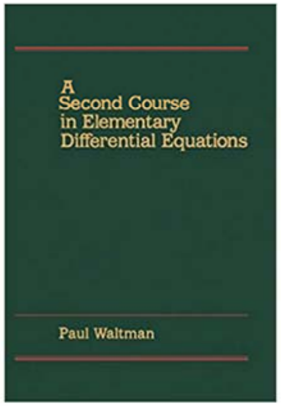
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.5. The Constant Coefficient Case: Real and Distinct Eigenvalues—Proofs of Theorems



Theorem 1.5.1

Theorem 1.5.1. Let A be a constant matrix. A fundamental matrix Φ for $y' = Ay$ is $\Phi = e^{At}$.

Proof. We have $\frac{d}{dt}[e^{At}] = Ae^{At}$ and $\det(e^{At}) \neq 0$ by Theorem 1.4.4. $\square$

Theorem 1.5.2

Theorem 1.5.2. If $B = TAT^{-1}$, then A and B have the same eigenvalues.

Proof. First, $B - \lambda\mathcal{I} = TAT^{-1} - \lambda TT^{-1} = T(A - \lambda\mathcal{I})T^{-1}$. By Theorem 1.2.3,

$$\det(B - \lambda\mathcal{I}) = \det(T)\det(A - \lambda\mathcal{I})\det(T^{-1}).$$

Since T (and T^{-1}) are invertible, then $\det(T) \neq 0$ and $\det(T^{-1}) \neq 0$ by Theorem 1.2.4. So $\det(B - \lambda\mathcal{I}) = 0$ if and only if $\det(A - \lambda\mathcal{I}) = 0$, so that A and B have the same eigenvalues, as claimed. $\square$

Theorem 1.5.3

Theorem 1.5.3. If A is a constant matrix, λ is an eigenvalue of A and $\vec{c}$ a corresponding eigenvector, then $\vec{y} = e^{\lambda t}\vec{c}$ is a solution of $\vec{y}\,' = A\vec{y}$.

Proof. First,

$$A\vec{y} = Ae^{\lambda t}\vec{c} = e^{\lambda t}(A\vec{c}) = e^{\lambda t}(\lambda\vec{c}) = \lambda e^{\lambda t}\vec{c}.$$

Secondly,

$$\vec{y}\,' = \frac{d}{dt}[e^{\lambda t}\vec{c}] = \lambda e^{\lambda t}\vec{c}.$$

So the claim holds. $\square$