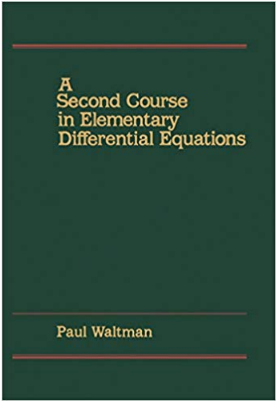
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.5. The Constant Coefficient Case: Real and Distinct Eigenvalues—Proofs of Theorems



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