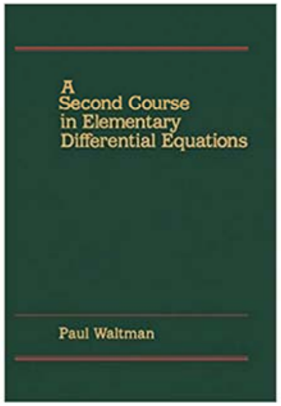
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.7. The Constant Coefficient Case: The Putzer Algorithm—Proofs of Theorems



Theorem 1.7.1. Putzer Algorithm

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Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$e^{At} = \sum_{j=0}^{n-1} R_{j+1}(t) P_j \qquad (7.1)$$

where $p_0 = \mathcal{I}$,

$$P_j = \prod_{k=1}^{j} (A - \lambda_k \mathcal{I}) \text{ for } j = 1, 2, \ldots, n-1 \qquad (7.2)$$

and $R_1(t), R_2(t), \ldots, R_n(t)$ is the solution of

$$\begin{cases} R_1' = \lambda_1 R_1 \\ R_j' = R_{j-1} + \lambda_j R_j, \ j = 2, 3, \ldots, n \\ R_1(0) = 1 \\ F_j(0) = 0, \ j = 2, 3, \ldots, n. \end{cases} \qquad (7.3)$$

Theorem 1.7.1 (continued 1)

Proof. Let $\Phi(t) = \sum_{j=0}^{n-1} R_{j+1}(t) P_j$. Define $R_0(t) \equiv 0$. Then by (7.3),

$$\Phi'(t) = \sum_{j=0}^{n-1} R_{j+1}'(t) P_j = \sum_{j=0}^{n-1} (\lambda_{j+1} R_{j+1}(t) + R_j(t)) P_j$$

so that

$$\begin{aligned} \Phi'(t) - \lambda_n \Phi(t) &= \sum_{j=0}^{n-1} (\lambda_{j+1} R_{j+1}(t) + R_j(t)) P_j - \lambda_n \sum_{j=0}^{n-1} R_j(t) P_j \\ &= \sum_{j=0}^{n-2} (\lambda_{j+1} - \lambda_n) R_{j+1}(t) P_j + \sum_{j=0}^{n-2} R_{j+1}(t) P_{j+1} \\ &= \sum_{j=0}^{n-2} \left((A - \lambda_{j+1} \mathcal{I}) P_j + (\lambda_{j+1} - \lambda_n) P_j \right) R_{j+1}(t) \text{ by (7.2)} \end{aligned}$$

Theorem 1.7.1 (continued 2)

Proof (continued).

$$\begin{aligned} \Phi'(t) - \lambda_n \Phi(t) &= \sum_{j=0}^{n-2} (A - \lambda_n \mathcal{I}) P_j R_{j+1}(t) = (A - \lambda_n \mathcal{I}) \sum_{j=0}^{n-2} P_j R_{j+1}(t) \\ &= (A - \lambda \mathcal{I}) \left(\sum_{j=0}^{n-1} P_j R_{j+1}(t) - P_{n-1} R_n(t) \right) \\ &= (A - \lambda_n \mathcal{I}) \left(\Phi(t) - R_n(t) P_{n-1} \right) \\ &= (A - \lambda_n \mathcal{I}) \left(\Phi - R_n(t) \prod_{k=1}^{n-1} (A - \lambda_k \mathcal{I}) \right) \\ &= (A - \lambda_n \mathcal{I}) \Phi(t) - R_n(t) \prod_{k=1}^{n} (A - \lambda_k \mathcal{I}) \text{ by (7.2)} \\ &= (A - \lambda_n \mathcal{I}) \Phi(t) - R_n(t) P_n \text{ by (7.2).} \qquad (*) \end{aligned}$$

Theorem 1.7.1 (continued 3)

Proof (continued). Now

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \prod_{k=1}^{n}(\lambda - \lambda_k)$$

and so $P_n = \prod_{k=1}^{n}(A - \lambda_k\mathcal{I}) = p(A)$. By the Cayley-Hamilton Theorem, $P_n = 0$. So

$$\Phi'(t) = \lambda_n\Phi(t) = (A - \lambda_n\mathcal{I})\Phi(t) \text{ from } (*)$$

and $\Phi'(t) = A\Phi(t)$. Also, $\Phi(0) = \sum_{j=0}^{n-1} R_{j+1}(0)P_j = P_0 = \mathcal{I}$. So if we consider the IVPs $\begin{cases} \vec{y}\,' = A\vec{y} \\ \vec{y}(0) = \vec{e}_i, \ \ i = 1, 2, \ldots, n \end{cases}$ we see that the ith column of Φ is a solution. By Theorem 1.3.1, solutions to IVPs are unique, so it follows that $\Phi(t) = e^{At}$. □