Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations
Section 1.7. The Constant Coefficient Case: The Putzer Algorithm
Proofs of Theorems

Theorem 1.7.1. Putzer Algorithm

Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Then

\[
e^{At} = \sum_{j=0}^{n-1} R_{j+1}(t)P_j
\]

(7.1)

where \( p_0 = I \),

\[
P_j = \prod_{k=1}^{j} (A - \lambda_k I) \quad \text{for} \quad j = 1, 2, \ldots, n-1
\]

(7.2)

and \( R_1(t), R_2(t), \ldots, R_n(t) \) is the solution of

\[
\begin{aligned}
R'_1 &= \lambda_1 R_1 \\
R'_j &= R_{j-1} + \lambda_j R_j, \quad j = 2, 3, \ldots, n \\
R_1(0) &= 1 \\
F_j(0) &= 0, \quad j = 2, 3, \ldots, n.
\end{aligned}
\]

(7.3)

### Theorem 1.7.1 (continued 1)

**Proof.** Let \( \Phi(t) = \sum_{j=0}^{n-1} R_{j+1}(t)P_j \). Define \( R_0(t) \equiv 0 \). Then by (7.3),

\[
\Phi'(t) = \sum_{j=0}^{n-1} R'_{j+1}(t)P_j = \sum_{j=0}^{n-1} (\lambda_{j+1} R_{j+1}(t) + R_j(t))P_j
\]

so that

\[
\Phi'(t) - \lambda_n \Phi(t) = \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_n) R_{j+1}(t)P_j + \sum_{j=0}^{n-2} R_j(t)P_j
\]

\[
= \sum_{j=0}^{n-2} R_j(t)P_j + \sum_{j=0}^{n-2} (A - \lambda_{j+1} I)P_j + (\lambda_{j+1} - \lambda_n)P_j R_{j+1}(t)
\]

by (7.2) and

\[
\phi(t) = \sum_{j=0}^{n-2} R_j(t)P_j + (A - \lambda_{n-1} I)P_n R_{n-1}(t)
\]

\[
= (A - \lambda_{n-1} I)P_n R_{n-1}(t)
\]

\[
= (A - \lambda_{n-1} I)P_n R_{n-1}(t)
\]

### Theorem 1.7.1 (continued 2)

**Proof (continued).**

\[
\Phi'(t) - \lambda_n \Phi(t) = \sum_{j=0}^{n-2} (A - \lambda_n I)P_j R_{j+1}(t) = (A - \lambda_n I) \sum_{j=0}^{n-2} R_j(t)P_j R_{j+1}(t)
\]

\[
= (A - \lambda I) \left( \sum_{j=0}^{n-1} P_j R_{j+1}(t) - P_{n-1} R_n(t) \right)
\]

\[
= (A - \lambda_n I) \left( (\Phi(t) - R_n(t)P_{n-1}) \right)
\]

\[
= (A - \lambda_n I) \left( \Phi(t) R_{n-1}(t) \prod_{k=1}^{n} (A - \lambda_k I) \right)
\]

\[
= (A - \lambda_n I) \Phi(t) R_{n-1}(t) \prod_{k=1}^{n} (A - \lambda_k I) \text{ by (7.2)}
\]

\[
= (A - \lambda_n I) \Phi(t) R_{n-1}(t) \text{ by (7.2).}
\]
Theorem 1.7.1 (continued 3)

Proof (continued). Now

\[ p(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \prod_{k=1}^{n} (\lambda - \lambda_k) \]

and so \( P_n = \prod_{k=1}^{n} (A - \lambda_k I) = p(A) \). By the Cayley-Hamilton Theorem, \( P_n = 0 \). So

\[ \Phi'(t) = \lambda_n \Phi(t) = (A - \lambda_n I) \Phi(t) \text{ from } (*) \]

and \( \Phi'(t) = A \Phi(t) \). Also, \( \Phi(0) = \sum_{j=0}^{n-1} R_{j+1}(0) P_j = P_0 = I \). So if we consider the IVPs

\[ \begin{cases} \dot{\vec{y}}' - A\vec{y} \\ \vec{y}(0) = \vec{e}_i, \quad i = 1, 2, \ldots, n \end{cases} \]

we see that the \( i \)th column of \( \Phi \) is a solution. By Theorem 1.3.1, solutions to IVPs are unique, so it follows that \( \Phi(t) = e^{At} \). \( \square \)