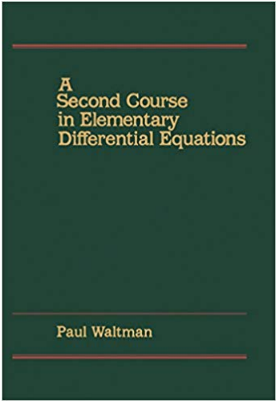
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.7. The Constant Coefficient Case: The Putzer Algorithm—Proofs of Theorems



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Theorem 1.7.1. Putzer Algorithm

Theorem 1.7.1. Putzer Algorithm.
Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$e^{At} = \sum_{j=0}^{n-1} R_{j+1}(t) P_j \qquad (7.1)$$

where $p_0 = \mathcal{I}$,

$$P_j = \prod_{k=1}^{j} (A - \lambda_k \mathcal{I}) \text{ for } j = 1, 2, \ldots, n-1 \qquad (7.2)$$

and $R_1(t), R_2(t), \ldots, R_n(t)$ is the solution of

$$\begin{cases} R_1' = \lambda_1 R_1 \\ R_j' = R_{j-1} + \lambda_j R_j, \ j = 2, 3, \ldots, n \\ R_1(0) = 1 \\ F_j(0) = 0, \ j = 2, 3, \ldots, n. \end{cases} \qquad (7.3)$$

Theorem 1.7.1 (continued 1)

Proof. Let $\Phi(t) = \sum_{j=0}^{n-1} R_{j+1}(t)P_j$. Define $R_0(t) \equiv 0$. Then by (7.3),

$$\Phi'(t) = \sum_{j=0}^{n-1} R'_{j+1}(t)P_j = \sum_{j=0}^{n-1}(\lambda_{j+1}R_{j+1}(t) + R_j(t))P_j$$

so that

$$\begin{aligned}
\Phi'(t) - \lambda_n\Phi(t) &= \sum_{j=0}^{n-1}(\lambda_{j+1}R_{j+1}(t) + R_j(t))P_j - \lambda_n \sum_{j=0}^{n-1} R_j(t)P_j \\
&= \sum_{j=0}^{n-2}(\lambda_{j+1} - \lambda_n)R_{j+1}(t)P_j + \sum_{j=0}^{n-2} R_{j+1}(t)P_{j+1} \\
&= \sum_{j=0}^{n-2}\left((A - \lambda_{j+1}\mathcal{I})P_j + (\lambda_{j+1} - \lambda_n)P_j\right) R_{j+1}(t) \text{ by (7.2)}
\end{aligned}$$

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Theorem 1.7.1 (continued 2)

Proof (continued).

$$\begin{aligned}
\Phi'(t) - \lambda_n\Phi(t) &= \sum_{j=0}^{n-2}(A - \lambda_n\mathcal{I})P_jR_{j+1}(t) = (A - \lambda_n\mathcal{I})\sum_{j=0}^{n-2}P_jR_{j+1}(t) \\
&= (A - \lambda\mathcal{I})\left(\sum_{j=0}^{n-1}P_jR_{j+1}(t) - P_{n-1}R_n(t)\right) \\
&= (A - \lambda_n\mathcal{I})\left(\Phi(t) - R_n(t)P_{n-1}\right) \\
&= (A - \lambda_n\mathcal{I})\left(\Phi - R_n(t)\prod_{k=1}^{n-1}(A - \lambda_k\mathcal{I})\right) \\
&= (A - \lambda_n\mathcal{I})\Phi(t) - R_n(t)\prod_{k=1}^{n}(A - \lambda_k\mathcal{I}) \text{ by (7.2)} \\
&= (A - \lambda_n\mathcal{I})\Phi(t) - R_n(t)P_n \text{ by (7.2).} \qquad (*)
\end{aligned}$$

Theorem 1.7.1 (continued 3)

Proof (continued). Now

$$p(\lambda) = \det(A - \lambda\mathcal{I}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \prod_{k=1}^{n}(\lambda - \lambda_k)$$

and so $P_n = \prod_{k=1}^{n}(A - \lambda_k\mathcal{I}) = p(A)$. By the Cayley-Hamilton Theorem, $P_n = 0$. So

$$\Phi'(t) = \lambda_n\Phi(t) = (A - \lambda_n\mathcal{I})\Phi(t) \text{ from } (*)$$

and $\Phi'(t) = A\Phi(t)$. Also, $\Phi(0) = \sum_{j=0}^{n-1} R_{j+1}(0)P_j = P_0 = \mathcal{I}$.

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