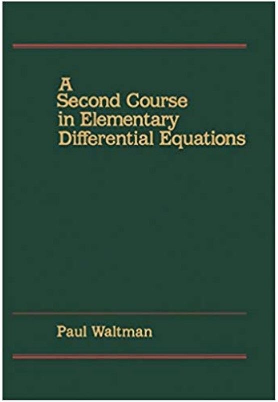
Advanced Differential Equations

Chapter 1. Systems of Linear Differential Equations

Section 1.8. General Linear Systems—Proofs of Theorems



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Theorem 1.8.1

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$$\vec{y}\,' - A\vec{y} = \vec{e}(t) \tag{8.2}$$

($\vec{e}(t)$ is called a *forcing term*), then any solution $\vec{\Psi}(t)$ of (8.2) can be written as

$$\vec{\Psi}(t) = \Phi(t)\vec{c} + \vec{x}(t)$$

where Φ is a fundamental matrix for

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and $\vec{c}$ is a constant.

Proof. Let $L(\vec{y}] = \vec{y}\,' = A\vec{y}$ and let $L(\vec{\Psi}(t)] = \vec{e}(t)$. Then $L[\vec{\Psi}(t) - \vec{x}(t)] = \vec{c}(t) - \vec{e}(t) = \vec{0}$. So $\vec{\Psi}(t) = \vec{x}(t)$ is a solution of (8.3). By Theorem 1.3.4, $\vec{\Psi}(t) - \vec{x}(t) = \Psi(t)\vec{c}$ and so $\vec{\Psi}(t) = \Psi(t)\vec{c} + \vec{x}(t)$. □

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$$\vec{x}(t) - \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)\vec{e}(\tau)\,d\tau$$

is a solution of $\vec{y}\,' = \vec{e}(t)$.

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$$\vec{x}\,'(t) = \Phi'(t)\int_{t_0}^{t} \Phi^{-1}(\tau)\vec{e}(\tau)\,d\tau + \Phi(t)\Psi^{-1}(t)\vec{e}(t).$$

Now $\Phi'(t) = A(t)\Phi(t)$ so

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