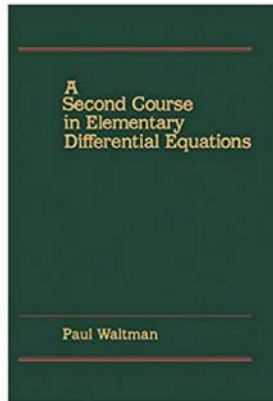


Advanced Differential Equations

Chapter 2. Two-Dimensional Autonomous Systems Section 2.2. The Phase Plane—Proofs of Theorems



Lemma 2.2.1

Lemma 2.2.1. If $(\varphi_1(t), \varphi_2(t))$ is a solution of (1.2), then so is $(\varphi_1(t - \tau), \varphi_2(t - \tau))$ for any $\tau \in \mathbb{R}$.

Proof. Let $\psi_i(t) = \varphi_i(t - \tau)$ for $i = 1, 2$. Then

$$\psi'_1(t) = \varphi'_1(t - \tau) = f(\varphi_1(t - \tau), \varphi_2(t - \tau)) = f(\psi_1(t), \psi_2(t))$$

and

$$\psi'_2(t) = \varphi'_2(t - \tau) = g(\varphi_1(t - \tau), \varphi_2(t - \tau)) = g(\psi_1(t), \psi_2(t))$$

and the result follows. \square

Theorem 2.2.2

Theorem 2.2.2. Let f and g be continuously differentiable. Through each point (x_0, y_0) there is a unique trajectory of
$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y). \end{aligned}$$

Proof. Suppose $(\varphi_1(t), \varphi_2(t))$ and $(\psi_1(t), \psi_2(t))$ pass through (x_0, y_0) ; i.e., $\varphi_1(t_0) = x_0 = \psi_1(t_1)$ and $\varphi_2(t_0) = y_0 = \psi_2(t_1)$. Since solutions to IVPs are unique, $(\varphi_1, \varphi_2) \neq (\psi_1, \psi_2)$ if and only if $t_0 \neq t_1$. By Lemma 2.2.1, $x_1(t) = \varphi_1(t - t_1 + t_0)$ and $x_2(t) = \varphi_2(t - t_1 + t_0)$ are a solution of (1.2). But $x_1(t_1) = \varphi_1(t_0) = x_0 = \psi_1(t_1)$ and $x_2(t_1) = \varphi_2(t_0) = y_0 = \psi_2(t_1)$. By Theorem 2.1.1 for IVPs, $x_1(t) = \psi_1(t)$ and $x_2(t) = \psi_2(t)$. But by Lemma 2.2.1, (x_1, x_2) and (φ_1, φ_2) determine the same trajectory. \square

Theorem 2.2.A

Theorem 2.2.A. The only trajectory which passes through a critical point (x_0, y_0) is the constant solution $x = x_0, y = y_0$.

Proof. The IVP
$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \\ x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}$$
 has a unique solution, so it must be the constant solution. \square