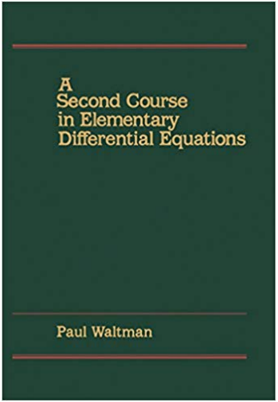
Advanced Differential Equations

Chapter 2. Two-Dimensional Autonomous Systems

Section 2.2. The Phase Plane—Proofs of Theorems



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Lemma 2.2.1

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Proof. Let $\psi_i(t) = \varphi_i(t-\tau)$ for $i = 1, 2$. Then

$$\psi_1'(t) = \varphi_1'(t-\tau) = f(\varphi_1(t-\tau), \varphi_2(t-\tau) = f(\psi_1(t), \psi_2(t))$$

and

$$\psi_2'(t) = \varphi_2'(t-\tau) = g(\varphi_1(t-\tau), \varphi_2(t-\tau) = g(\psi_1(t), \psi_2(t))$$

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Theorem 2.2.2

Theorem 2.2.2. Let f and g be continuously differentiable. Through each point (x_0, y_0) there is a unique trajectory of $\begin{array}{l} x' = f(x,y) \\ y' = g(x,y). \end{array}$

Proof. Suppose $(\varphi_1(t), \varphi_2(t))$ and $(\psi_1(t), \psi_2(t))$ pass through (x_0, y_0); i.e., $\varphi(t_0) = x_0 = \psi_1(t_1)$ and $\varphi_2(t_0) = y_0 = \psi_2(t_1)$. Since solutions to IVPs are unique, $(\varphi_1, \varphi_2) \neq (\psi_1, \psi_2)$ if and only if $t_0 \neq t_1$. By Lemma 2.2.1, $x_1(t) = \varphi_1(t - t_1 + t_0)$ and $x_2(t) = \varphi_2(t - t_1 + t_0)$ are a solution of (1.2).

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Theorem 2.2.2

Theorem 2.2.2. Let f and g be continuously differentiable. Through each point (x_0, y_0) there is a unique trajectory of $\begin{matrix} x' = f(x,y) \\ y' = g(x,y). \end{matrix}$

Proof. Suppose $(\varphi_1(t), \varphi_2(t))$ and $(\psi_1(t), \psi_2(t))$ pass through (x_0, y_0); i.e., $\varphi(t_0) = x_0 = \psi_1(t_1)$ and $\varphi_2(t_0) = y_0 = \psi_2(t_1)$. Since solutions to IVPs are unique, $(\varphi_1, \varphi_2) \neq (\psi_1, \psi_2)$ if and only if $t_0 \neq t_1$. By Lemma 2.2.1, $x_1(t) = \varphi_1(t - t_1 + t_0)$ and $x_2(t) = \varphi_2(t - t_1 + t_0)$ are a solution of (1.2). But $x_1(t_1) = \varphi_1(t_0) = x_0 = \psi_1(t_1)$ and
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Theorem 2.2.A

Theorem 2.2.A. The only trajectory which passes through a critical point (x_0, y_0) is the constant solution $x = x_0, y = y_0$.

Proof. The IVP $\begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{array}$ has a unique solution, so it must be the constant solution. □

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