Chapter 2. Two-Dimensional Autonomous Systems

Section 2.4. Critical Points of General 2-D Linear Systems—Proofs of Theorems
Example. Exercise 2.2.4
Example. (Exercise 2.4.4.) If $D = T^{-1}AT$ is a diagonal matrix, then the columns of $T$ are eigenvectors of $T$ are eigenvectors of $A$ and the nonzero entries of $D$ are eigenvalues of $A$.

Proof. Suppose $D = T^{-1}AT$ where $D$ is diagonal. Then $AT = TD$. Suppose $T = [t_{ij}]$ and $D = [d_{ij}]$. Then the $(i, j)$ entry of $TD$ is

$$
\sum_{k=1}^{n} t_{ik}d_{kj} = t_{ij}d_{jj}.
$$

So the $j$th column of $TD$ is $d_{jj} \begin{bmatrix} t_{1j} \\ t_{2j} \\ \vdots \\ t_{nj} \end{bmatrix}$. Therefore, the $j$th column of $T$ is an eigenvector of $A$ with eigenvalue $d_{jj}$.
Example. (Exercise 2.4.4.) If $D = T^{-1} AT$ is a diagonal matrix, then the columns of $T$ are eigenvectors of $T$ are eigenvectors of $A$ and the nonzero entries of $D$ are eigenvalues of $A$.

Proof. Suppose $D = T^{-1} AT$ where $D$ is diagonal. Then $AT = TD$. Suppose $T = [t_{ij}]$ and $D = [d_{ij}]$. Then the $(i, j)$ entry of $TD$ is

$$
\sum_{k=1}^{n} t_{ik} d_{kj} = t_{ij} d_{jj}.
$$

So the $j$th column of $TD$ is $d_{jj}$ times the $j$th column of $T$. Therefore, the $j$th column of $T$ is an eigenvector of $A$ with eigenvalue $d_{jj}$.
Example. (Exercise 2.4.4.) If \( D = T^{-1}AT \) is a diagonal matrix, then the columns of \( T \) are eigenvectors of \( T \) are eigenvectors of \( A \) and the nonzero entries of \( D \) are eigenvalues of \( A \).

**Proof.** Suppose \( D = T^{-1}AT \) where \( D \) is diagonal. Then \( AT = TD \).

Suppose \( T = [t_{ij}] \) and \( D = [d_{ij}] \). Then the \((i, j)\) entry of \( TD \) is

\[
\sum_{k=1}^{n} t_{ik}d_{kj} = t_{ij}d_{jj}.
\]

So the \( j \)th column of \( TD \) is \( d_{jj} \)
\[
\begin{bmatrix}
t_{1j} \\
t_{2j} \\
\vdots \\
t_{nj}
\end{bmatrix}
\]

That is, the \( j \)th column of \( TD \) is \( d_{jj} \) times the \( j \)th column of \( T \). So the \( j \)th column of \( AT \) is \( d_{jj} \) times the \( j \)th column of \( T \). Therefore, the \( j \)th column of \( T \) is an eigenvector of \( A \) with eigenvalue \( d_{jj} \). \( \square \)