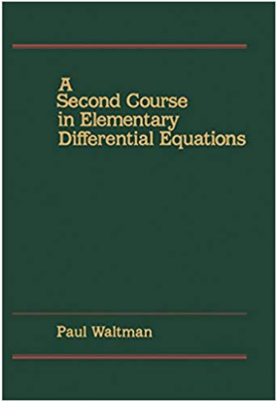
Advanced Differential Equations

Chapter 3. Existence Theory

Section 3.2. Preliminaries—Proofs of Theorems



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Theorem 3.2.1

Theorem 3.2.1. Let $f(t, y)$ be continuous. A function $\varphi(t)$ defined on interval I is a solution to the IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

on I if and only if φ is a continuous solution of the integral equation

$$y = y_0 + \int_{t_0}^{t} f(s, y)\, ds.$$

Proof. Let φ be a solution of the IVP and let $K(t) = f(t, \varphi(t))$. Then $K(t)$ is continuous and $\varphi'(t) = K(t)$. So $\int_{t_0}^{t} \varphi'\, ds = \int_{t_0}^{t} K(s)\, ds$ or $\varphi(t) - \varphi(t_0) = \int_{t_0}^{t} f(s, \varphi(s))\, ds$ which implies $\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds$. So φ is a solution of the integral equation.

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Theorem 3.2.1 (continued)

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Proof (continued). If φ is a solution of the integral equation then $\varphi(t) = y_0 + \int_{t_0}^{t} f(s,\varphi(s))\,ds$ and so $\varphi(t_0) = y_0$. Also since f and φ are continuous, then $f(t,\varphi(t))$ is continuous and so by the Fundamental Theorem of Calculus, $\varphi'(t) = f(t,\varphi(t))$. So φ is a solution to the IVP. □

Theorem 3.2.2

Theorem 3.2.2. The space $C([a, b])$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C([a, b])$. For a given t, $\{f_n(t)\}$ is a sequence of real numbers and since

$$|f_n(t) = f_m(t)| \leq \max_{t \in [a,b]} |f_n(t) = f_m(t)| = \rho(f_n, f_m),$$

$\{f_n(t)\}$ is a Cauchy sequence of real numbers and so converges because $\mathbb{R}$ is complete. Define the function $y(t) = \lim_{n \to \infty} f_n(t)$ for each $t \in [a, b]$.

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Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $\rho(f_n, f_m) < \varepsilon$ for all $m, n > N$. Then for any $t \in [a,b]$, $|f_n(t) - f_m(t) \leq \rho(f_n, f_m) < \varepsilon$ for all $m, n > N$. Therefore $\lim_{m\to\infty} |f_n(t) - f_m(t)| < \varepsilon$ or

$$|f_n(t) - y(t)| < \varepsilon \text{ for all } n > N \text{ and for all } t \in [a,b]. \qquad (*)$$

If $y \in C([a,b])$, then this implies $\rho(f_n, y) < \varepsilon$ and we get that f_n converges to y.

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Theorem 3.2.2 (continued)

Proof (continued). We now show $y \in C([a,b])$ (i.e., y is continuous). Let $\varepsilon > 0$ and $t_0 \in [a,b]$ be fixed. As above, there exists $N \in \mathbb{N}$ such that $|f_n(t) - y(t)| < \varepsilon/3$ for all $n > N$ and for all $y \in [a,b]$ by $(*)$. Fix $n > N$. Since $f_n(x)$ is continuous, there exists $\delta > 0$ such that if $|t - t_0| < \delta$ then $|f_n(t) - f_n(t_0)| < \varepsilon/3$. So

$$|y(t) - y(t_0)| \le |y(t) - f_n(t)| + |f_n(t) - f_n(t_0)| + |f_n(t_0) - y(t_0)|$$

and if $|t - t_0| < \delta$ then

$$|y(t) - y(t_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So $y(t)$ is continuous at t_0 and $y \in C([a,b])$. □

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Lemma 3.2.3

Lemma 3.2.3. The space B is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in B. As in Theorem 3.2.2, f_n converges to $y \in C([a, b])$.

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Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\rho(f_n, y) < \varepsilon$. So

$$\rho(y, f_0) \leq \rho(y, f_n) + \rho(f_n, f_0) < \varepsilon + \alpha.$$

Therefore $\rho(y, f_0) \leq \alpha$ and $y \in B$. $\square$

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