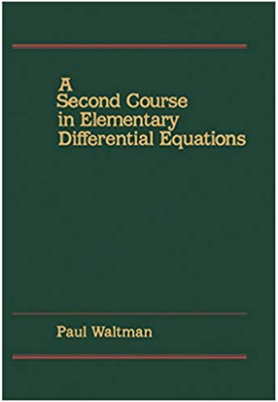
Advanced Differential Equations

Chapter 3. Existence Theory

Section 3.3. The Contraction Mapping Theorem—Proofs of Theorems



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Proof. Let $y_0 \in M$ and define $y_n = Ty_{n-1}$ for $n \in \mathbb{N}$. Then

$$\rho(y_{m+1}, y_m) = \rho(Ty_m, Ty_{m-1}) \leq \alpha\rho(y_m, y_{m-1}) \leq \cdots \leq \alpha^m \rho(y_1, y_0).$$

Then for $k \in \mathbb{N}$ we have

$$\begin{aligned} \rho(y_{n+k}, y_n) &\leq \rho(y_{n+k}, y_{n+k-1}) + \cdots + \rho(y_{n+1}, y_n) \\ &\leq \alpha^{n+k-1}\rho(y_1, y_0) + \cdots \alpha^n \rho(y_1, y_0) \\ &= \alpha^n(\alpha^{k-1} + \alpha^{k-2} + \cdots \alpha + 1)\rho(y_1, y_0). \end{aligned}$$

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Since for $\alpha \in (0,1)$, $\sum_{n=1}^{\infty} \alpha^n = 1/(1-\alpha)$, and therefore $\sum_{i=0}^{k-1} \alpha^i < 1/(1-\alpha)$. So

$$\rho(y_{n+k}, y_n) \leq \frac{\alpha^n}{1-\alpha}\rho(y_1, y_0).$$

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Theorem 3.3.1 (continued)

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Proof. Since $\alpha \in (0,1)$, $\dfrac{\alpha^n}{1-\alpha}\rho(y_1, y_0) \to 0$ as $n \to \infty$, then $\{y_n\}$ is Cauchy. Since M is complete, $\{y_n\}$ converges to, say, y. Not T is continuous (take $\delta = \varepsilon$ in the definition of continuity) and so

$$Ty = T\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} y_{n+1} = y.$$

So y is a fixed point.

Now suppose $x = Tx$ and $y = Ty$. Then, since $\alpha \in (0,1)$,

$$\rho(x,y) = \rho(Tx, Ty) \leq \alpha\rho(x,y) \text{ implies } \rho(x,y) = 0.$$

So fixed points are unique. □

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