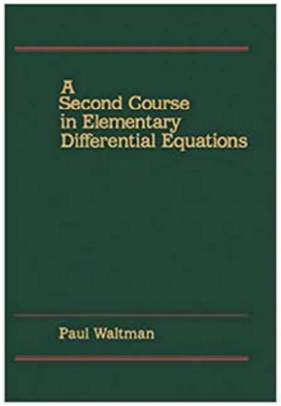
Advanced Differential Equations

Chapter 3. Existence Theory

Section 3.4. The IVP for One Scalar Differential Equations—Proofs of Theorems



Theorem 3.4.1

Theorem 3.4.1. Let $f(x,t)$ be continuous and Lipschitz with Lipschitz constant K on $\Omega = \{(t,y) \mid |t - t_0| \le a, |y - y_0| \le b\}$ and let M be a number such that $|f(t,y)| \le M$ for all $(x,y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

for $|t - t_0| \le \alpha$.

Proof. Let f, Ω, t_0, a, b and α be as hypothesized. Let $B = \{\varphi \mid \varphi \in C([t_0 - \alpha, t_0 + \alpha]), \rho(\varphi, y_0) \le b\}$. Then B is complete by Lemma 3.2.3. Define

$$T[\varphi](t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds.$$

Now
$$|(T\varphi)(t) - y_0| \le \left| \int_{t_0}^{t} |f(s,\varphi(s))|\, ds \right| \le M|t - t_0| \le M\alpha < b.$$

Theorem 3.4.1 (continued)

Proof (continued). Since $|(t\varphi)(t) - y_0|$ is continuous, it attains its maximum on $[t_0 - \alpha, t_0 + \alpha]$ (by the Extreme Value Theorem) and this max is $\le b$. So $\rho(T\varphi, y_0) \le b$ and for each $\varphi \in B$ we have $T\varphi \in B$. That is, $T : B \to B$.

Now

$$\begin{aligned} |(T\varphi - T\psi)(t)| &= \left| y_0 + \int_{t_0}^{t} f(s,\varphi(s))\, ds - y_0 - \int_{t_0}^{t} f(s,\psi(s))\, ds \right| \\ &\le \left| \int_{t_0}^{t} |f(s,\varpi(s)) - f(s,\psi(s))|\, ds \right| \le K \left| \int_{t_0}^{t} |\varphi(s) - \psi(s)|\, ds \right| \\ &\le K\rho(\varphi,\psi) \left| \int_{t_0}^{t} ds \right| = K\rho(\varphi,\psi)|t - t_0| \le K\alpha\rho(\varphi,\psi). \end{aligned}$$

So $|(T\varphi - T\psi)(t)| \le K\alpha\rho(\varphi,\psi)$ and therefore

$$\rho(T\varphi, T\psi) \le K\alpha\rho(\varphi,\psi) < \rho(\varphi,\psi).$$

Theorem 3.4.1 (continued)

Theorem 3.4.1. Let $f(x,t)$ be continuous and Lipschitz with Lipschitz constant K on $\Omega = \{(t,y) \mid |t - t_0| \le a, |y - y_0| \le b\}$ and let M be a number such that $|f(t,y)| \le M$ for all $(x,y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

for $|t - t_0| \le \alpha$.

Proof. So T is a contraction on B and therefore T has a unique fixed point by Theorem 3.3.1. This fixed point ψ is a solution to the integral equation

$$y = y_0 + \int_{t_0}^{t} f(s, \psi(s))\, ds.$$

By Theorem 3.2.1, this fixed point is also the unique solution to the IVP. □

Theorem 3.4.2

Theorem 3.4.2. Let $f(t,y)$ be continuous and Lipschitz with Lipschitz constant K valid for every t and y (i.e., f is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

and this solution is valid for all t.

Proof. Let $t_1 \in \mathbb{R}$ and assume $t_1 > t_0$. Let $M = C([t_0, t_1])$. For $f, g \in M$, define

$$\rho(f,g) = \max_{t \in [t_0,t_1]} e^{-L(t-t_0)}|f(t) - g(t)|.$$

Then ρ is a metric on M. We restrict $L > K$. Define

$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Then $T : M \to M$.

Theorem 3.4.2 (continued 1)

Proof (continued). If $g, h \in M$ then

$$|(Tg)(t) - (Th)(t)| = \left| \int_{t_0}^{t} (f(s, g(s)) - f(s, h(s)))\, ds \right|$$

$$\leq \left| \int_{t_0}^{t} |f(s,g(s)) - f(s,h(s))|\, ds \right| \leq K \left| \int_{t_0}^{t} |g(s) - h(s)|\, ds \right|$$

$$= K \int_{t_0}^{t} |g(s) - h(s)|\, ds,$$

or, with $\alpha = K/L < 1$,

$$e^{-L(t-t_0)}|(Tg)(t) - (Th)(t)| \leq K e^{-L(t-t_0)} \int_{t_0}^{t} e^{-L(s-t_0)} e^{L(s-t_0)} |g(s) - h(s)|\, ds$$

$$\leq K\rho(g,h) e^{-L(t-t_0)} \int_{t_0}^{t} e^{L(s-t_0)}\, ds = \frac{K}{L}\rho(g,h) e^{-L(t-t_0)} (e^{L(t-t_0)} - 1)$$

$$= \frac{K}{L}\rho(g,h)(1 - e^{-L(t-t_0)}) < \frac{K}{L}\rho(g,h) = \alpha\rho(g,h).$$

Theorem 3.4.2 (continued 2)

Theorem 3.4.2. Let $f(t,y)$ be continuous and Lipschitz with Lipschitz constant K valid for every t and y (i.e., f is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

and this solution is valid for all t.

Proof (continued). So $\rho(Tg, Th) \leq \alpha\rho(g,h)$ and T is a contraction. As in Theorem 3.4.1, this produces a unique fixed solution of the IVP valid on $[t_0, t_1]$. Since t_1 was arbitrary, the solution is valid for $t \in [t_0, \infty)$. Similarly, if $t_1 < t_0$ we get the solution valid for all t. □

Lemma 3.4.4. Gronwall's Inequality

Lemma 3.4.4. Gronwall's Inequality.
Let $\varphi(t)$ be a nonnegative function where

$$\varphi(t) \leq C + K \int_{t_0}^{t} \varphi(s)\, ds, \quad t > t_0$$

where C and K are constants, $K \geq 0$ and $C > 0$. Then $\varphi(t) \leq Ce^{K(t-t_0)}$ for $t > t_0$.

Proof. Under the stated hypotheses, $\dfrac{K\varphi(t)}{C + K\int_{t_0}^{t} \varphi(s)\, ds} \leq K$ and so

$$\int_{u=t_0}^{u=t} \frac{K\varphi(u)}{C + K\int_{t_0}^{u} \varphi(s)\, ds}\, du \leq \int_{u=t_0}^{u=t} K\, du$$

$$\text{or } \ln\left(C + K \int_{t_0}^{u} \varphi(s)\, ds \right)\Bigg|_{u=t_0}^{u=t} \leq K(t - t_0)$$

Lemma 3.4.4 (continued)

Proof (continued).

$$\text{or } \ln\left(C + \int_{t_0}^{t} \varphi(s)\,ds\right) = \ln C \leq K(t - t_0)$$

$$\text{or } \frac{C + K\int_{t_0}^{t} \varphi(s)\,ds}{C} \leq e^{K(t-t_0)}$$

$$\text{or } C + K\int_{t_0}^{t} \varphi(s)\,ds \leq Ce^{K(t-t_0)}$$

and so by the hypotheses,

$$\varphi(t) \leq C + K\int_{t_0}^{t} \varphi(s)\,ds \leq Ce^{K(t-t_0)}.$$

□

Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions

Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions.
Define $T : \mathbb{R} \to C([a,b])$ be defined as $Ty_0 = \varphi$ where φ is the solution of

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

for given Lipschitz f with Lipschitz constant K valid for every t and y. Then T is continuous.

Proof. Let $\varepsilon > 0$ and choose $\delta < \varepsilon/e^{K(b-a)}$. By Theorem 3.2.1, $Ty_0 = \varphi$ is equivalent to

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds.$$

Let $y_0, y_1 \in \mathbb{R}$ where $|y_0 - y_1| < \delta$. Then if $Ty_1 = \psi$, we have...

Theorem 3.4.5 (continued)

Proof (continued).

$$|Ty_0 - Ty_1| = |\varphi(t) - \psi(t)|$$

$$= \left|\left(y_0 + \int_{t_0}^{t} f(s,\varphi(s))\,ds\right) - \left(y_1 + \int_{t_0}^{t} f(s,\psi(s))\,ds\right)\right|$$

$$= \left|y_0 - y_1 + \int_{t_0}^{t} (f(s,\varphi(s)) - f(s,\psi(s))\,ds\right|$$

$$\leq |y_0 + y_1| + \left|\int_{t_0}^{t} |f(s,\varphi(s)) - f(s,\psi(s))|\,ds\right|$$

$$\leq |y_0 - y_1| + K\left|\int_{t_0}^{t} |\varphi(s) - \psi(s)|\,ds\right| \qquad (*)$$

since K is the Lipschitz constant for f. By Lemma 3.4.4, we see that $(*)$ implies $|Ty_0 - Ty_1| \leq |y_0 - y_1|e^{K|t_0 - t|} \leq |y_0 - y_1|e^{K(b-a)} < \varepsilon$ for all $t \in [a,b]$. So $\rho(Ty_0, Ty_1) < \varepsilon$ and T is continuous. □