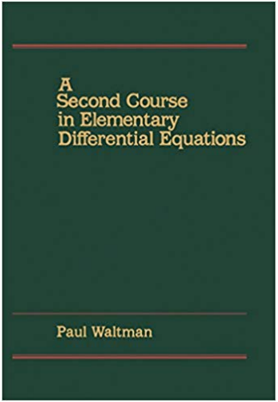
# Advanced Differential Equations

## Chapter 3. Existence Theory

Section 3.4. The IVP for One Scalar Differential Equations—Proofs of Theorems



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# Theorem 3.4.1

**Theorem 3.4.1.** Let $f(x,t)$ be continuous and Lipschitz with Lipschitz constant $K$ on $\Omega = \{(t,y) \mid |t-t_0| \le a, |y-y_0| \le b\}$ and let $M$ be a number such that $|f(t,y)| \le M$ for all $(x,y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

for $|t-t_0| \le \alpha$.

**Proof.** Let $f, \Omega, t_0, a, b$ and $\alpha$ be as hypothesized. Let $B = \{\varphi \mid \varphi \in C([t_0-\alpha, t_0+\alpha]), \rho(\varphi, y_0) \le b\}$. Then $B$ is complete by Lemma 3.2.3. Define

$$T[\varphi](t) = y_0 + \int_{t_0}^{t} f(s,\varphi(s))\,ds.$$

Now
$$|(T\varphi)(t) - y_0| \le \left| \int_{t_0}^{t} |f(s,\varphi(s))|\,ds \right| \le M|t-t_0| \le M\alpha < b.$$

# Theorem 3.4.1

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$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

for $|t-t_0| \leq \alpha$.

**Proof.** Let $f, \Omega, t_0, a, b$ and $\alpha$ be as hypothesized. Let $B = \{\varphi \mid \varphi \in C([t_0-\alpha, t_0+\alpha]), \rho(\varphi, y_0) \leq b\}$. Then $B$ is complete by Lemma 3.2.3. Define

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Now
$$|(T\varphi)(t) - y_0| \leq \left| \int_{t_0}^{t} |f(s,\varphi(s))|\, ds \right| \leq M|t-t_0| \leq M\alpha < b.$$

# Theorem 3.4.1 (continued)

**Proof (continued).** Since $|(t\varphi)(t) - y_0|$ is continuous, it attains its maximum on $[t_0 - \alpha, t_0 + \alpha]$ (by the Extreme Value Theorem) and this max is $\leq b$. So $\rho(T\varphi, y_0) \leq b$ and for each $\varphi \in B$ we have $T\varphi \in B$. That is, $T : B \to B$.

Now

$$\begin{aligned} |(T\varphi - T\psi)(t)| &= \left| y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds - y_0 - \int_{t_0}^{t} f(s, \psi(s))\, ds \right| \\ &\leq \left| \int_{t_0}^{t} |f(s, \varpi(s)) - f(s, \psi(s))|\, ds \right| \leq K \left| \int_{t_0}^{t} |\varphi(s) - \psi(s)|\, ds \right| \\ &\leq K\rho(\varphi, \psi) \left| \int_{t_0}^{t} ds \right| = K\rho(\varphi, \psi)|t - t_0| \leq K\alpha\rho(\varphi, \psi). \end{aligned}$$

So $|(T\varphi - T\psi)(t)| \leq K\alpha\rho(\varphi, \psi)$ and therefore

$$\rho(T\varphi, T\psi) \leq K\alpha\rho(\varphi, \psi) < \rho(\varphi, \psi).$$

# Theorem 3.4.1 (continued)

**Proof (continued).** Since $|(t\varphi)(t) - y_0|$ is continuous, it attains its maximum on $[t_0 - \alpha, t_0 + \alpha]$ (by the Extreme Value Theorem) and this max is $\leq b$. So $\rho(T\varphi, y_0) \leq b$ and for each $\varphi \in B$ we have $T\varphi \in B$. That is, $T : B \to B$.

Now

$$|(T\varphi - T\psi)(t)| = \left| y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds - y_0 - \int_{t_0}^{t} f(s, \psi(s))\, ds \right|$$

$$\leq \left| \int_{t_0}^{t} |f(s, \varpi(s)) - f(s, \psi(s))|\, ds \right| \leq K \left| \int_{t_0}^{t} |\varphi(s) - \psi(s)|\, ds \right|$$

$$\leq K\rho(\varphi, \psi) \left| \int_{t_0}^{t} ds \right| = K\rho(\varphi, \psi)|t - t_0| \leq K\alpha\rho(\varphi, \psi).$$

So $|(T\varphi - T\psi)(t)| \leq K\alpha\rho(\varphi, \psi)$ and therefore

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# Theorem 3.4.1 (continued)

**Theorem 3.4.1.** Let $f(x,t)$ be continuous and Lipschitz with Lipschitz constant $K$ on $\Omega = \{(t,y) \mid |t-t_0| \leq a, |y-y_0| \leq b\}$ and let $M$ be a number such that $|f(t,y)| \leq M$ for all $(x,y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

for $|t-t_0| \leq \alpha$.

**Proof.** So $T$ is a contraction on $B$ and therefore $T$ has a unique fixed point by Theorem 3.3.1. This fixed point $\psi$ is a solution to the integral equation

$$y = y_0 + \int_{t_0}^{t} f(s,\psi(s))\,ds.$$

By Theorem 3.2.1, this fixed point is also the unique solution to the IVP. □

# Theorem 3.4.2

**Theorem 3.4.2.** Let $f(t, y)$ be continuous and Lipschitz with Lipschitz constant $K$ valid for every $t$ and $y$ (i.e., $f$ is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

and this solution is valid for all $t$.

**Proof.** Let $t_1 \in \mathbb{R}$ and assume $t_1 > t_0$. Let $M = C([t_0, t_1])$. For $f, g \in M$, define

$$\rho(f, g) = \max_{t \in [t_0, t_1]} e^{-L(t-t_0)} |f(t) - g(t)|.$$

Then $\rho$ is a metric on $M$. We restrict $L > K$. Define

$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Then $T : M \to M$.

# Theorem 3.4.2

**Theorem 3.4.2.** Let $f(t,y)$ be continuous and Lipschitz with Lipschitz constant $K$ valid for every $t$ and $y$ (i.e., $f$ is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

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$$(Ty)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Then $T : M \to M$.

# Theorem 3.4.2 (continued 1)

**Proof (continued).** If $g, h \in M$ then

$$|(Tg)(t) - (Th)(t)| = \left| \int_{t_0}^{t} (f(s, g(s)) - f(s, h(s)))\, ds \right|$$

$$\leq \left| \int_{t_0}^{t} |f(s, g(s)) - f(s, h(s))|\, ds \right| \leq K \left| \int_{t_0}^{t} |g(s) - h(s)|\, ds \right|$$

$$= K \int_{t_0}^{t} |g(s) - h(s)|\, ds,$$

or, with $\alpha = K/L < 1$,

$$e^{-L(t-t_0)}|(Tg)(t) - (Th)(t)| \leq K e^{-L(t-t_0)} \int_{t_0}^{t} e^{-L(s-t_0)} e^{L(s-t_0)} |g(s) - h(s)|\, ds$$

$$\leq K\rho(g, h) e^{-L(t-t_0)} \int_{t_0}^{t} e^{L(s-t_0)}\, ds = \frac{K}{L}\rho(g, h) e^{-L(t-t_0)}(e^{L(t-t_0)} - 1)$$

$$= \frac{K}{L}\rho(g, h)(1 - e^{-L(t-t_0)}) < \frac{K}{L}\rho(g, h) = \alpha\rho(g, h).$$

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$$\leq K \rho(g, h) e^{-L(t-t_0)} \int_{t_0}^{t} e^{L(s-t_0)}\, ds = \frac{K}{L} \rho(g, h) e^{-L(t-t_0)} (e^{L(t-t_0)} - 1)$$

$$= \frac{K}{L} \rho(g, h)(1 - e^{-L(t-t_0)}) < \frac{K}{L} \rho(g, h) = \alpha \rho(g, h).$$

# Theorem 3.4.2 (continued 2)

**Theorem 3.4.2.** Let $f(t, y)$ be continuous and Lipschitz with Lipschitz constant $K$ valid for every $t$ and $y$ (i.e., $f$ is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

and this solution is valid for all $t$.

**Proof (continued).** So $\rho(Tg, Th) \leq \alpha\rho(g, h)$ and $T$ is a contraction. As in Theorem 3.4.1, this produces a unique fixed solution of the IVP valid on $[t_0, t_1]$. Since $t_1$ was arbitrary, the solution is valid for $t \in [t_0, \infty)$. Similarly, if $t_1 < t_0$ we get the solution valid for all $t$. □

# Lemma 3.4.4. Gronwall's Inequality

**Lemma 3.4.4. Gronwall's Inequality.**
Let $\varphi(t)$ be a nonnegative function where

$$\varphi(t) \leq C + K \int_{t_0}^{t} \varphi(s)\, ds, \ \ t > t_0$$

where $C$ and $K$ are constants, $K \geq 0$ and $C > 0$. Then $\varphi(t) \leq Ce^{K(t-t_0)}$ for $t > t_0$.

**Proof.** Under the stated hypotheses, $\dfrac{K\varphi(t)}{C + K\int_{t_0}^{t} \varphi(s)\, ds} \leq K$ and so

$$\int_{u=t_0}^{u=t} \frac{K\varphi(u)}{C + K\int_{t_0}^{u} \varphi(s)\, ds}\, du \leq \int_{u=t_0}^{u=t} K\, du$$

$$\text{or } \ln\left(C + K\int_{t_0}^{u} \varphi(s)\, ds\right)\Bigg|_{u=t_0}^{u=t} \leq K(t - t_0)$$

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$$\text{or } \ln\left(C + K\int_{t_0}^{u} \varphi(s)\, ds\right)\Bigg|_{u=t_0}^{u=t} \leq K(t - t_0)$$

# Lemma 3.4.4 (continued)

**Proof (continued).**

$$\text{or } \ln\left(C + \int_{t_0}^{t} \varphi(s)\,ds\right) = \ln C \le K(t - t_0)$$

$$\text{or } \frac{C + K\int_{t_0}^{t} \varphi(s)\,ds}{C} \le e^{K(t-t_0)}$$

$$\text{or } C + K\int_{t_0}^{t} \varphi(s)\,ds \le Ce^{K(t-t_0)}$$

and so by the hypotheses,

$$\varphi(t) \le C + K\int_{t_0}^{t} \varphi(s)\,ds \le Ce^{K(t-t_0)}.$$

□

# Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions

**Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions.**
Define $T : \mathbb{R} \to C([a, b])$ be defined as $Ty_0 = \varphi$ where $\varphi$ is the solution of

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

for given Lipschitz $f$ with Lipschitz constant $K$ valid for every $t$ and $y$. Then $T$ is continuous.

**Proof.** Let $\varepsilon > 0$ and choose $\delta < \varepsilon / e^{K(b-a)}$. By Theorem 3.2.1, $Ty_0 = \varphi$ is equivalent to

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds.$$

Let $y_0, y_1 \in \mathbb{R}$ where $|y_0 - y_1| < \delta$. Then if $Ty_1 = \psi$, we have...

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$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds.$$

Let $y_0, y_1 \in \mathbb{R}$ where $|y_0 - y_1| < \delta$. Then if $Ty_1 = \psi$, we have...

# Theorem 3.4.5 (continued)

**Proof (continued).**

$$|Ty_0 - Ty_1| = |\varphi(t) - \psi(t)|$$

$$= \left| \left( y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds \right) - \left( y_1 + \int_{t_0}^{t} f(s, \psi(s))\, ds \right) \right|$$

$$= \left| y_0 - y_1 + \int_{t_0}^{t} (f(s, \varphi(s)) - f(s, \psi(s))\, ds \right|$$

$$\leq |y_0 + y_1| + \left| \int_{t_0}^{t} |f(s, \varphi(s)) - f(s, \psi(s))|\, ds \right|$$

$$\leq |y_0 - y_1| + K \left| \int_{t_0}^{t} |\varphi(s) - \psi(s)|\, ds \right| \qquad (*)$$

since $K$ is the Lipschitz constant for $f$. By Lemma 3.4.4, we see that $(*)$ implies $|Ty_0 - Ty_1| \leq |y_0 - y_1| e^{K|t_0 - t|} \leq |y_0 - y_1| e^{K(b-a)} < \varepsilon$ for all $t \in [a, b]$. So $\rho(Ty_0, Ty_1) < \varepsilon$ and $T$ is continuous. □

# Theorem 3.4.5 (continued)

**Proof (continued).**

$$|Ty_0 - Ty_1| = |\varphi(t) - \psi(t)|$$

$$= \left| \left( y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds \right) - \left( y_1 + \int_{t_0}^{t} f(s, \psi(s))\, ds \right) \right|$$

$$= \left| y_0 - y_1 + \int_{t_0}^{t} (f(s, \varphi(s)) - f(s, \psi(s))\, ds \right|$$

$$\leq |y_0 + y_1| + \left| \int_{t_0}^{t} |f(s, \varphi(s)) - f(s, \psi(s))|\, ds \right|$$

$$\leq |y_0 - y_1| + K \left| \int_{t_0}^{t} |\varphi(s) - \psi(s)|\, ds \right| \qquad (*)$$

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