APPLIED MATH - LIAPUNOV FUNCTIONS

Liapunov functions can be thought of as expressing the potential energy of a system. Extrema of potential energy in a system such as a pendulum correspond to equilibria of the system. Liapunov functions can be used to analyze equilibria of a system of differential equations to determine stability.

definition. Let V be defined on some domain D (that is, D is an open connected set) of $\mathbf{R}^2$ containing $(0,0)$. V is *positive definite* on D if $V(0,0) = 0$ and $V(x,y) > 0$ for all other $(x,y) \in D$. If $V(0,0) = 0$ and $V(x,y) \geq 0$ for all $(x,y) \in D$, then V is *positive semidefinite.* Negative definite and negative semidefinite are similarly defined.

definition. Consider the autonomous system:

$$(*)\begin{cases} dx/dt &= F(x,y) \\ dy/dt &= G(x,y) \end{cases}$$

and a function $V(x,y)$. Define $\dot{V} = V_x(x,y)F(x,y) + V_y(x,y)G(x,y)$. $\dot{V}$ is the *derivative of V with respect to the system* $(*)$.

Note. $\dot{V}$ is the rate of change of V along a trajectory of $(*)$ that passes through the point (x,y). We now present uses of these functions V to determine stability and instability of critical points. The following theorems are from *Introduction to Differential Equations* by Boyce and DiPrima.

Theorem 9.6.1 Suppose $(*)$ has an isolated critical point at $(0,0)$. If there exists a V that is continuous and has continuous first partial derivatives, is positive definite and $\dot{V}$ is negative definite on some domain D in $\mathbf{R}^2$ containing $(0,0)$ then $(0,0)$ is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite, then $(0,0)$ is a stable critical point.

Theorem 9.6.2 Suppose $(*)$ has an isolated critical point at $(0,0)$. Let V be continuous with continuous first partial derivatives. Suppose $V(0,0) = 0$ and that in every neigborhood of $(0,0)$ there is a point at which V is positive (negative). Then if there is a domain D containing $(0,0)$ such that $\dot{V}$ is positive definite (negative definite) on D, then $(0,0)$ is an unstable critical point.

definition. A function V satisfying the conditions of Theorem 9.6.1 or 9.6.2 is called a *Liapunov function.*

Note. We also introduce the idea of a "basin of attraction."

Theorem 9.6.3 Let $(0,0)$ be an isolated critical point of $(*)$. Let V be continuous and have continuous first partial derivatives. If there is a bounded domain D_K containing $(0,0)$ where $V(x,y) < K$, V is positive definite, and $\dot{V}$ is negative defninite, then every solution that starts in D_K approaches $(0,0)$ as $t \to \infty$. That is, D_K is in the "basin of attraction" of $(0,0)$.

definition. The Liénard equation is

$$y'' + f(y)y' + h(y) = 0,$$

where $f(y) > 0$ and $yh(y) > 0, y \neq 0$ and f and h continuous.

Problem 2.6.3(a) Analyze the Liénard equation using Liapunov functions.
Solution. Notice that since $yh(y) > 0$ for $y \neq 0$, then $h(y) < 0$ for $y < 0$ and $h(y) > 0$ for $y > 0$. Since h is continuous, $h(0) = 0$. Converting the DE to a system (in terms of x and y) gives:

$$\begin{aligned} dx/dt &= y &&= F(x,y) \\ dy/dt &= -f(x)y - h(x) &&= G(x,y). \end{aligned}$$

As on page 139, we look for a Liapunov function as follows. Define

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x h(s)\,ds.$$

Then $V(0,0) = 0$ and $V(x,y) > 0$ for $(x,y) \neq (0,0)$, i.e. V is positive definite. Now

$$\dot{V}(x,y) = h(x)y + y(-f(x)y - h(x)) = -y^2 f(x).$$

Since $f(x) > 0$, we see that $\dot{V}$ is negative semidefinite. Therefore by Theorem 9.6.1, $(0,0)$ is a stable critical point for the DE.

Problem 2.6.4 Modify the results of Problem 2.6.3 to apply to the damped pendulum equation $y'' + y' + \sin(y) = 0$.
Solution. As a system, the damped pendulum equation is

$$\begin{aligned} x' &= y \\ y' &= -y - \sin(x). \end{aligned}$$

Therefore this system has critical points of the form $(k\pi, 0)$ where $k \in \mathbf{Z}$.

At $(0,0)$ we get, as above,

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x \sin(s)\,ds = \frac{1}{2}y^2 + 1 - \cos(x).$$

and

$$\dot{V}(x,y) = \sin(x)y + y(-y - \sin(x)) = -y^2.$$

Therefore V is positive definite "near $(0,0)$" (that is, in a domain avoiding points (x,y) with $y = 0$ and x an even multiple of π), and $\dot{V}$ is negative semidefinite. So by Theorem 9.6.1, $(0,0)$ is a stable critical point.

At $(\pi, 0)$, we translate the system to

$$\begin{aligned} x' &= y \\ y' &= -y - \sin(x - \pi). \end{aligned}$$

As above, let

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x \sin(y-\pi) = \frac{1}{2}y^2 - 1 - \cos(x-\pi).$$

Then $V(0,0) = 0$ and $V(x,y) < 0$ for (x,y) in some domain of $(0,0)$. Therfore V is negative definite. Also, $\dot{V}(x,y) = -y^2$ and so $\dot{V}$ is negative semidefinite. Theorem 9.6.2 implies that $(0,0)$ is an unstable critical point of this system and $(\pi,0)$ is an unstable critical point of the damped pendulum equation. (Actually, to apply Theorem 9.6.2, we need $\dot{V}$ to be negative definite, which is the case if we omit points of the form $(x,0)$. This tells us that these critical points are actually saddle points.)

Of course, in the damped pendulum equation, critical points of the form $(2k\pi,0)$ behave as does the critical point $(0,0)$ and critical points of the from $((2k+1)\pi,0)$ behave as does the critical point $(\pi,0)$.