Liapunov functions can be thought of as expressing the potential energy of a system. Extrema of potential energy in a system such as a pendulum correspond to equilibria of the system. Liapunov functions can be used to analyze equilibria of a system of differential equations to determine stability.

**Definition.** Let \( V \) be defined on some domain \( D \) (that is, \( D \) is an open connected set) of \( \mathbb{R}^2 \) containing \((0,0)\). \( V \) is **positive definite** on \( D \) if \( V(0,0) = 0 \) and \( V(x,y) > 0 \) for all other \((x,y) \in D \). If \( V(0,0) = 0 \) and \( V(x,y) \geq 0 \) for all \((x,y) \in D \), then \( V \) is **positive semidefinite**. Negative definite and negative semidefinite are similarly defined.

**Definition.** Consider the autonomous system:

\[
\begin{align*}
\frac{dx}{dt} &= F(x,y) \\
\frac{dy}{dt} &= G(x,y)
\end{align*}
\]

and a function \( V(x,y) \). Define \( \dot{V} = V_x(x,y)F(x,y) + V_y(x,y)G(x,y) \). \( \dot{V} \) is the derivative of \( V \) with respect to the system (\( \ast \)).

**Note.** \( \dot{V} \) is the rate of change of \( V \) along a trajectory of (\( \ast \)) that passes through the point \((x,y)\).

We now present uses of these functions \( V \) to determine stability and instability of critical points. The following theorems are from *Introduction to Differential Equations* by Boyce and DiPrima.

**Theorem 9.6.1** Suppose (\( \ast \)) has an isolated critical point at \((0,0)\). If there exists a \( V \) that is continuous and has continuous first partial derivatives, is positive definite and \( \dot{V} \) is negative definite on some domain \( D \) in \( \mathbb{R}^2 \) containing \((0,0)\) then \((0,0)\) is an asymptotically stable critical point. If \( \dot{V} \) is negative semidefinite, then \((0,0)\) is a stable critical point.

**Theorem 9.6.2** Suppose (\( \ast \)) has an isolated critical point at \((0,0)\). Let \( V \) be continuous with continuous first partial derivatives. Suppose \( V(0,0) = 0 \) and that in every neighborhood of \((0,0)\) there is a point at which \( V \) is positive (negative). Then if there is a domain \( D \) containing \((0,0)\) such that \( \dot{V} \) is positive definite (negative definite) on \( D \), then \((0,0)\) is an unstable critical point.

**Definition.** A function \( V \) satisfying the conditions of Theorem 9.6.1 or 9.6.2 is called a Liapunov function.

**Note.** We also introduce the idea of a "basin of attraction."

**Theorem 9.6.3** Let \((0,0)\) be an isolated critical point of (\( \ast \)). Let \( V \) be continuous and have continuous first partial derivatives. If there is a bounded domain \( D_K \) containing \((0,0)\) where \( V(x,y) < K \), \( V \) is positive definite, and \( \dot{V} \) is negative definite, then every solution that starts in \( D_K \) approaches \((0,0)\) as \( t \to \infty \). That is, \( D_K \) is in the "basin of attraction" of \((0,0)\).
The Liénard equation is
\[ y'' + f(y)y' + h(y) = 0, \]
where \( f(y) > 0 \) and \( yh(y) > 0, y \neq 0 \) and \( f \) and \( h \) continuous.

**Problem 2.6.3** Analyze the Liénard equation using Liapunov functions.

**Solution.** Notice that since \( yh(y) > 0 \) for \( y \neq 0 \), then \( h(y) < 0 \) for \( y < 0 \) and \( h(y) > 0 \) for \( y > 0 \).
Since \( h \) is continuous, \( h(0) = 0 \). Converting the DE to a system (in terms of \( x \) and \( y \)) gives:
\[
\begin{align*}
\frac{dx}{dt} &= y &= F(x, y) \\
\frac{dy}{dt} &= -f(x)y - h(x) &= G(x, y).
\end{align*}
\]
As on page 139, we look for a Liapunov function as follows. Define
\[ V(x, y) = \frac{1}{2}y^2 + \int_0^x h(s) \, ds. \]
Then \( V(0, 0) = 0 \) and \( V(x, y) > 0 \) for \( (x, y) \neq (0, 0) \), i.e. \( V \) is positive definite. Now
\[ \dot{V}(x, y) = h(x)y + y(-f(x)y - h(x)) = -y^2f(x). \]
Since \( f(x) > 0 \), we see that \( \dot{V} \) is negative semidefinite. Therefore by Theorem 9.6.1, \((0, 0)\) is a stable critical point for the DE.

**Problem 2.6.4** Modify the results of Problem 2.6.3 to apply to the damped pendulum equation
\[ y'' + y' + \sin(y) = 0. \]

**Solution.** As a system, the damped pendulum equation is
\[
\begin{align*}
x' &= y \\
y' &= -y - \sin(x).
\end{align*}
\]
Therefore this system has critical points of the form \((k\pi, 0)\) where \( k \in \mathbb{Z} \).
At \((0, 0)\) we get, as above,
\[ V(x, y) = \frac{1}{2}y^2 + \int_0^x \sin(s) \, ds = \frac{1}{2}y^2 + 1 - \cos(x). \]
and
\[ \dot{V}(x, y) = \sin(x)y + y(-y - \sin(x)) = -y^2. \]
Therefore \( V \) is positive definite “near \((0, 0)\)” (that is, in a domain avoiding points \((x, y)\) with \( y = 0 \) and \( x \) an even multiple of \( \pi \)), and \( \dot{V} \) is negative semidefinite. So by Theorem 9.6.1, \((0, 0)\) is a stable critical point.
At \((\pi, 0)\), we translate the system to
\[
\begin{align*}
x' &= y \\
y' &= -y - \sin(x - \pi).
\end{align*}
\]
As above, let

\[ V(x, y) = \frac{1}{2} y^2 + \int_0^x \sin(y - \pi) \, dy \]

Then \( V(0, 0) = 0 \) and \( V(x, y) < 0 \) for \((x, y)\) in some domain of \((0, 0)\). Therefore \( V \) is negative definite. Also, \( \dot{V}(x, y) = -y^2 \) and so \( V \) is negative semidefinite. Theorem 9.6.2 implies that \((0, 0)\) is an unstable critical point of this system and \((\pi, 0)\) is an unstable critical point of the damped pendulum equation. (Actually, to apply Theorem 9.6.2, we need \( \dot{V} \) to be negative definite, which is the case if we omit points of the form \((x, 0)\). This tells us that these critical points are actually saddle points.)

Of course, in the damped pendulum equation, critical points of the form \((2k\pi, 0)\) behave as does the critical point \((0, 0)\) and critical points of the from \(((2k + 1)\pi, 0)\) behave as does the critical point \((\pi, 0)\).