Applied Math I - FALL 1996

CHAPTER 1 SECTION 2 ANSWERS

2. A matrix is said to be *diagonal* if $a_{ij} = 0$ when $i \neq j$. The product of two diagonal matrices is diagonal.

   Proof. Let the matrices be $A = [a_{ij}]$ and $B = [b_{ij}]$ (with appropriate dimensions). Let $AB = [c_{ij}]$. Then if $i \neq j$,

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{p} a_{ik} b_{kj} \\ &= a_{ii} b_{ij} \text{ since } a_{ik} = 0 \text{ for } i \neq k \\ &= 0 \text{ since } b_{ij} = 0 \text{ for } i \neq j. \end{aligned}$$

   Therefore AB is diagonal. ∎

6. For all matrices A and B of the appropriate size, $(AB)^T = B^T A^T$.

   Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $AB = [c_{ij}]$. Then

$$AB = [c_{ij}] = \left[\sum_{k=1}^{p} a_{ik} b_{kj} \right]$$

   and

$$(AB)^T = [c'_{ij}] = [c_{ij}] = \left[\sum_{k=1}^{p} a_{jk} b_{ki} \right].$$

   Now $A^T = [a'_{ij}] = [a_{ji}]$ and $B^T = [b'_{ij}] = [a_{ji}]$, so

$$B^T A^T = \left[\sum_{k=1}^{p} b'_{ik} a'_{kj} \right] = \left[\sum_{k=1}^{p} b_{ki} a_{jk} \right] = \left[\sum_{k=1}^{p} a_{jk} b_{ki} \right] = (AB)^T.$$

   ∎

Problem. Prove that an nth order linear DE can be written as a system of n linear first order DEs.

Proof. A system of n linear first order DEs looks like:

$$\begin{aligned} y_1' &= a_{11}(t) y_1 + a_{12}(t) y_2 + \cdots + a_{1n}(t) y_n + e_1(t) \\ y_2' &= a_{21}(t) y_1 + a_{22}(t) y_2 + \cdots + a_{2n}(t) y_n + e_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t) y_1 + a_{n2}(t) y_2 + \cdots + a_{nn}(t) y_n + e_n(t). \end{aligned}$$

Consider the nth order linear DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = e(t).$$

Let

$$\begin{aligned} y_1 &= y \\ y_2 &= y_1' = y' \\ y_3 &= y_2' = y'' \\ &\vdots \\ y_{n-1} &= y_{n-2}' = y^{(n-2)} \\ y_n &= y_{n-1}' = y^{(n-1)} \end{aligned}$$

Notice that $y_n' = y^{(n)}$ and therefore we can rewrite this nth order DE as the system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= \frac{-1}{a_n(t)}\left\{a_0(t)y_1 + a_1(t)y_2 + \cdots + a_{n-2}(t)y_{n-1} + a_{n-1}(t)y_n + e(t)\right\}. \end{aligned}$$

∎