1.4.2 Prove that if \( \Phi(t) \) and \( \Psi(t) \) are fundamental matrices for \( x' = Ax \), then there is a constant, nonsingular matrix \( C \) such that \( \Phi(t)C = \Psi(t) \).

**Proof.** Since \( \Psi \) is a fundamental matrix, the \( i \)-th column of \( \Psi \) is a solution to \( x' = Ax \) for \( i \in \{1, 2, \ldots, n\} \). Therefore, by Theorem 3.4, there exists a constant (column) vector \( c_i \) such that the \( i \)-th column of \( \Psi \) equals \( \Phi c_i \) (since \( \Phi \) is a fundamental matrix). Hence \( \Psi \) can be expressed as the product \( \Phi C \) where \( C = [c_i] \).

Now \( \det(\Phi C) = \det(\Phi) \det(C) = \det(\Psi) \). Since \( \Psi \) is a fundamental matrix, its columns are linearly independent and therefore it is nonsingular by Theorem 2.6. Since \( \det(\Phi) \det(C) = \det(\Psi) \), we see that \( \det(C) \neq 0 \) and therefore \( C \) is nonsingular.

1.3.4(a). Define an operator \( Tx \) on the set of continuous functions on \([0, 1]\) by
\[
(Tx)(t) = \int_0^t x(s) \, ds, \quad t \in [0, 1].
\]
Show that \( T \) is a linear operator.

**Proof.** We have
\[
(T(\alpha x + \beta y))(t) = \int_0^t (\alpha x(s) + \beta y(s)) \, ds
\]
\[
= \int_0^t \alpha x(s) \, ds + \int_0^t \beta y(s) \, ds
\]
\[
= \alpha \int_0^t x(s) \, ds + \beta \int_0^t y(s) \, ds
\]
\[
= \alpha (T(x))(t) + \beta (T(y))(t).
\]

Therefore, by definition, \( T \) is linear.

1.3.4(b). This operator can be defined on a larger class of functions.

**Proof.** Consider the class of bounded functions which are continuous on \([0, 1/2] \cap (1/2, 1]\). (We consider bounded functions to avoid integrals which might be infinite.) Then for \( t \in [0, 1/2] \), the calculations of part (a) show that \( T \) is linear. \( T \) is seen to be linear for \( t \in (1/2, 1] \) as in part (a) since
\[
\int_0^t x(s) \, ds = \int_0^{1/2} x(s) \, ds + \int_{1/2}^t x(s) \, ds
\]
(recall that integrals over discontinuities are defined in terms of limits). A similar argument shows that this operator can in fact be extended to the class of bounded functions defined on \([0, 1]\) where each function has a finite number of discontinuities.