Applied Math I - FALL 1996
CHAPTER 1 SECTION 4 ANSWERS

1.4.2 Prove that if $\Phi(t)$ and $\Psi(t)$ are fundamental matrices for $x' = Ax$, then there is a constant, nonsingular matrix C such that $\Phi(t)C = \Psi(t)$.

Proof. Since Ψ is a fundamental matrix, the ith column of Ψ is a solution to $\mathbf{x}' = A\mathbf{x}$ for $i \in \{1, 2, \ldots, n\}$. Therefore, by Theorem 3.4, there exists a constant (column) vector $\mathbf{c}_i$ such that the ith column of Ψ equals $\Phi\mathbf{c}_i$ (since Φ is a fundamental matrix). Hence Ψ can be expressed as the product ΦC where $C = [\mathbf{c}_i]$.

Now $\det(\Phi C) = \det(\Phi)\det(C) = \det(\Psi)$. Since Ψ is a fundamental matrix, its columns are linearly independent and therefore it is nonsingular by Theorem 2.6. Since $\det(\Phi)\det(C) = \det(\Psi)$, we see that $\det(C) \neq 0$ and therefore C is nonsingular. ∎

1.3.4(a). Define an operator Tx on the set of continuous functions on $[0, 1]$ by $(Tx)(t) = \int_0^t x(s)\,ds$, $t \in [0, 1]$. Show that T is a linear operator.

Proof. We have

$$\begin{aligned}
(T(\alpha x + \beta y))(t) &= \int_0^t (\alpha x(s) + \beta y(s))\,ds \\
&= \int_0^t \alpha x(s)\,ds + \int_0^t \beta y(s)\,ds \\
&= \alpha \int_0^t x(s)\,ds + \beta \int_0^t y(s)\,ds \\
&= \alpha(T(x))(t) + \beta(T(y))(t).
\end{aligned}$$

Therefore, by definition, T is linear. ∎

1.3.4(b). This operator can be defined on a larger class of functions.

Proof. Consider the class of bounded functions which are continuous on $[0, 1/2) \cap (1/2, 1]$. (We consider bounded functions to avoid integrals which might be infinite.) Then for $t \in [0, 1/2]$, the calculations of part (a) show that T is linear. T is seen to be linear for $t \in (1/2, 1]$ as in part (a) since $\int_0^t x(s)\,ds = \int_0^{1/2} x(s)\,ds + \int_{1/2}^t x(s)\,ds$ (recall that integrals over discontinuities are defined in terms of limits). A similar argument shows that this operator can in fact be extended to the class of bounded functions defined on $[0, 1]$ where each function has a finite number of discontinuities. ∎