APPLIED MATH 1 TEST 2 - Fall 1996

NAME ______________________________ STUDENT NUMBER ________________________

1. Answer each of the following.

(a) State an existence and uniqueness theorem for the system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y). \end{aligned}$$

Be sure to include all hypotheses.

(b) What does it mean for the point (x_0, y_0) to be a *stable* critical point? What does it mean for the point (x_0, y_0) to be an *asymptotically stable* critical point?

(c) Draw the trajectories for which the critical point (x_0, y_0) is a

i. node,

ii. saddle point,

iii. center,

iv. asymptotically stable spiral point, and

v. unstable critical point.

2. **(a)** In the system $\mathbf{x}' = A\mathbf{x}$ where A is 2×2, what can be said about the eigenvalues of A if $(0,0)$ is a

i. node,

ii. saddle point,

iii. center,

iv. asymptotically stable spiral point, and

v. unstable critical point.

(b) State a theorem relating the behavior of $(0,0)$ as a critical point for the two systems

$$\begin{aligned} x' &= ax + by \qquad & z_1' &= az_1 + bz_2 + \epsilon_1(z_1, z_2) \\ y' &= cx + dy \qquad & z_2' &= cz_1 + dz_2 + \epsilon_2(z_1, z_2) \end{aligned}$$

where $\epsilon_i(z_1, z_2)$ is continuously differentiable, $\epsilon_i(0,0) = 0$, and $\partial \epsilon_i / \partial z_i(0,0) = 0$ for $i \in \{1, 2\}$.

(c) Suppose

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y). \end{aligned}$$

has an isolated critical point at $(0,0)$. What conditions must a function V (a Liapunov function) satisfy to insure $(0,0)$ is an asympotically stable critical point? Include a definition of $\dot{V}$.

3. Do **each** of the following.

(a) Diagonalize the matrix

$$A = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}.$$

That is, find T, T^{-1} and diagonal matrix D such that $D = T^{-1}AT$.

(b) Use the results of part (a) to solve the system $\mathbf{x}' = A\mathbf{x}$ by considering the system $\mathbf{x}' = D\mathbf{x}$.

4. Find the critical points of the system

$$\begin{aligned} x' &= y \\ y' &= x - y + x(x - 2y). \end{aligned}$$

Find the linearization of the system at $(0,0)$ and discuss the stability or instability of $(0,0)$ as a critical point of the original nonlinear system.

5. Use $V(x,y) = x^2 + y^2$ to analyze the system

$$\begin{aligned} x' &= -y \\ y' &= x + y^5 - 2y. \end{aligned}$$

Include all relevant discussions of "positive definite," etc.

Bonus. Discuss the stabilty of all critical points (other than $(0,0)$) for the nonlinear system given in problem 4.