APPLIED MATH 1 TEST 2 - Fall 1996

NAME ___________________________ STUDENT NUMBER ___________________________

1. Answer each of the following.

   (a) State an existence and uniqueness theorem for the system

   \[
   \begin{align*}
   x' &= f(x, y) \\
   y' &= g(x, y).
   \end{align*}
   \]

   Be sure to include all hypotheses.

   (b) What does it mean for the point \((x_0, y_0)\) to be a stable critical point? What does it mean for the point \((x_0, y_0)\) to be an asymptotically stable critical point?

   (c) Draw the trajectories for which the critical point \((x_0, y_0)\) is a

   i. node,
   ii. saddle point,
   iii. center,
   iv. asymptotically stable spiral point, and
   v. unstable critical point.

2. (a) In the system \(\mathbf{x}' = A\mathbf{x}\) where \(A\) is \(2 \times 2\), what can be said about the eigenvalues of \(A\) if \((0,0)\) is a

   i. node,
   ii. saddle point,
   iii. center,
   iv. asymptotically stable spiral point, and
   v. unstable critical point.

   (b) State a theorem relating the behavior of \((0,0)\) as a critical point for the two systems

   \[
   \begin{align*}
   x' &= ax + by \\
   y' &= cx + dy \\
   z'_1 &= az_1 + bz_2 + \varepsilon_1(z_1, z_2) \\
   z'_2 &= cz_1 + dz_2 + \varepsilon_2(z_1, z_2)
   \end{align*}
   \]

   where \(\varepsilon_i(z_1, z_2)\) is continuously differentiable, \(\varepsilon_i(0,0) = 0\), and \(\partial\varepsilon_i/\partial z_i(0,0) = 0\) for \(i \in \{1,2\}\).

   (c) Suppose

   \[
   \begin{align*}
   x' &= f(x, y) \\
   y' &= g(x, y).
   \end{align*}
   \]

   has an isolated critical point at \((0,0)\). What conditions must a function \(V\) (a Liapunov function) satisfy to insure \((0,0)\) is an asymptotically stable critical point? Include a definition of \(V\).
3. Do each of the following.

(a) Diagonalize the matrix

\[ A = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}. \]

That is, find \( T, \ T^{-1} \) and diagonal matrix \( D \) such that \( D = T^{-1}AT \).

(b) Use the results of part (a) to solve the system \( x' = Ax \) by considering the system \( x' = Dx \).

4. Find the critical points of the system

\[
\begin{align*}
x' &= y \\
y' &= x - y + x(x - 2y).
\end{align*}
\]

Find the linearization of the system at \((0,0)\) and discuss the stability or instability of \((0,0)\) as a critical point of the original nonlinear system.

5. Use \( V(x,y) = x^2 + y^2 \) to analyze the system

\[
\begin{align*}
x' &= -y \\
y' &= x + y^5 - 2y.
\end{align*}
\]

Include all relevant discussions of "positive definite," etc.

**Bonus.** Discuss the stability of all critical points (other than \((0,0)\)) for the nonlinear system given in problem 4.