# Chapter 1. Systems of Linear Differential Equations

## Section 1.1. Introduction

**Note.** In this section we give some preliminary definitions and examples.

**Definition.** An *ordinary differential equation* ("ODE") is a relation among an independent variable $x$, an (unknown) function $y(x)$ of that variable, and certain of its derivatives. More precisely, an ODE is a function of $n+2$ variables evaluated at the independent variable $t$, the function $y$, and the $n$ derivatives of $y$, set equal to a constant:

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0.$$

A function $y$ satisfying this equation is a *solution*. The highest order of derivative of $y$ appearing in the equation is the *order* of the ODE.

**Example.** $\cos ty = y'$ is an ODE of order 1 (and I don't know the solutions).

**Example.** $y'' = -y$ is an ODE of order 2 and $y = \cos t$ and $y = \sin t$ are solutions (so is $y = a_1 \cos t + a_2 \sin t$).

**Note.** Solving a DE is, in some crude sense, closely related to integration. Recall from calculus that is is easy to differentiate, but hard to integrate. Therefore (!) DEs must, in general, be hard to solve.

**Definition.** A *system of DEs* is an equation of the form

$$F_i(x, y_1, y_1', \dots, y_1^{(v_1)}; y_2, y_2', \dots, y_2^{(v_2)}; \dots; y_m, y_m', \dots, y_n^{(v_m)}) = 0$$

for $i = 1, 2, \dots, m$. The *order* of the system of $v_1 + v_2 + \cdots + v_m$.

**Example.** $\begin{cases} \tan(ty_1')y_2'' + 7 = 0 \\ \cos y_2 + e^{y_1'} + t = 0 \end{cases}$ is a system of order 3 (notice that the number of equations equals the number of unknown functions).

**Definition.** A DE is said to be of *normal type* if it can be put in the form

$$y_i^{(v_i)} = f_i(x, y_1, y_2', \dots, y_1^{(v_1-1)}; y_2, y_2', \dots, y_2^{(v_2-1)}; \dots; y_m, y_m', \dots, y_m^{v_m-1)})$$

for $i = 1, 2, \dots, m$.

**Example.** $\begin{cases} y_1' = \cos t \\ y_2' = y_1 \\ y_3' = y_2 \end{cases}$ is a system of DEs of normal type, involving 3 equations (and 3 unknown functions) of order 3.

**Note.** A solution to the previous DE is $\begin{cases} y_1 = \sin x \\ y_2 = -\cos x \\ y_3 = -\sin x. \end{cases}$ Also, this system of 3 equations is equivalent to the 3rd oder DE: $y_3''' = \cos x$.

**Note.** In this class, we will only be concerned with systems of DEs of normal type.

**Note.** In Chapter 1, we are only concerned with systems of first order normal type DEs:

$$y_i' = f_i(t, y_1, y_2, \ldots, y_n) \text{ for } i = 1, 2, \ldots, n. \qquad (*)$$

In fact, we are even more restrictive than this.

**Definition.** A function $f$ is *linear* if $f(a_1t_1 + a_2t_2) = a_1f(t_1) + a_2f(t_2)$ for all constants $a_1, a_2$ and variables $t_1, t_2$.

**Note.** A continuous function $f(t)$ is linear if and only if it is of the form $f(t) = at$ for some constant $a$.

**Note.** If each $f_i$ in $(*)$ is linear in the variables $y_1, y_2, \ldots, y_n$ (but not necessarily in the first variable; $t$), then $(*)$ can be written as

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + e_1(t) \\ y_2' &= a_{21}y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + e_2(t) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + e_n(t). \end{aligned}$$

Notice the “matrix nature” of this equation.

**Definition.** An $n$th order DE is *linear* if it is of the form

$$a_n(t)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = e(t).$$

**Homework.** Prove that an $n$th order linear DE can be written as a system of $n$ linear first order DEs (see page 2 of the book).

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