Chapter 1. Systems of Linear Differential Equations

Section 1.1. Introduction

**Note.** In this section we give some preliminary definitions and examples.

**Definition.** An *ordinary differential equation* ("ODE") is a relation among an independent variable $x$, an (unknown) function $y(x)$ of that variable, and certain of its derivatives. More precisely, an ODE is a function of $n+2$ variables evaluated at the independent variable $t$, the function $y$, and the $n$ derivatives of $y$, set equal to a constant:

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0.$$ 

A function $y$ satisfying this equation is a *solution*. The highest order of derivative of $y$ appearing in the equation is the *order* of the ODE.

**Example.** $\cos ty = y'$ is an ODE of order 1 (and I don’t know the solutions).

**Example.** $y'' = -y$ is an ODE of order 2 and $y = \cos t$ and $y = \sin t$ are solutions (so is $y = a_1 \cos t + a_2 \sin t$).
Note. Solving a DE is, in some crude sense, closely related to integration. Recall from calculus that is is easy to differentiate, but hard to integrate. Therefore (!) DEs must, in general, be hard to solve.

Definition. A system of DEs is an equation of the form

$$F_i(x, y_1, y'_1, \ldots, y_{(v_1)}^{(1)}; y_2, y'_2, \ldots, y_{(v_2)}^{(1)}; \ldots; y_m, y'_m, \ldots, y_{(v_m)}^{(n)}) = 0$$

for $i = 1, 2, \ldots, m$. The order of the system of $v_1 + v_2 + \cdots + v_m$.

Example. \[
\begin{cases}
    \tan(ty_1) y_2'' + 7 = 0 \\
    \cos y_2 + e^{y_1} + t = 0
\end{cases}
\]
is a system of order 3 (notice that the number of equations equals the number of unknown functions).

Definition. A DE is said to be of normal type if it can be put in the form

$$y_i^{(v_i)} = f_i(x, y_1, y'_1, \ldots, y_{(v_1-1)}^{(1)}; y_2, y'_2, \ldots, y_{(v_2-1)}^{(1)}; \ldots; y_m, y'_m, \ldots, y_{(v_m-1)}^{(n-1)})$$

for $i = 1, 2, \ldots, m$.

Example. \[
\begin{cases}
    y'_1 = \cos t \\
    y'_2 = y_1 \\
    y'_3 = y_2
\end{cases}
\]
is a system of DEs of normal type, involving 3 equations (and 3 unknown functions) of order 3.
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Note. A solution to the previous DE is

\[
\begin{align*}
y_1 &= \sin x \\
y_2 &= -\cos x \\
y_3 &= -\sin x.
\end{align*}
\]

Also, this system of 3 equations is equivalent to the 3rd order DE: \( y'''_3 = \cos x \).

Note. In this class, we will only be concerned with systems of DEs of normal type.

Note. In Chapter 1, we are only concerned with systems of first order normal type DEs:

\[
y'_i = f_i(t, y_1, y_2, \ldots, y_n) \quad \text{for } i = 1, 2, \ldots, n. \quad (\ast)
\]

In fact, we are even more restrictive than this.

Definition. A function \( f \) is linear if \( f(a_1 t_1 + a_2 t_2) = a_1 f(t_1) + a_2 f(t_2) \) for all constants \( a_1, a_2 \) and variables \( t_1, t_2 \).

Note. A continuous function \( f(t) \) is linear if and only if it is of the form \( f(t) = at \) for some constant \( a \).

Note. If each \( f_i \) in \((\ast)\) is linear in the variables \( y_1, y_2, \ldots, y_n \) (but not necessarily in the first variable; \( t \)), then \((\ast)\) can be written as

\[
\begin{align*}
y'_1 &= a_{11} y_1 + a_{12}(t) y_2 + \cdots + a_{1n}(t) y_n + e_1(t) \\
y'_2 &= a_{21} y_1 + a_{22}(t) y_2 + \cdots + a_{2n}(t) y_n + e_2(t) \\
&\vdots \\
y'_n &= a_{n1} y_1 + a_{n2}(t) y_2 + \cdots + a_{nn}(t) y_n + e_n(t).
\end{align*}
\]

Notice the “matrix nature” of this equation.
Definition. An $n$th order DE is linear if it is of the form

$$a_n(t)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = e(t).$$

Homework. Prove that an $n$th order linear DE can be written as a system of $n$ linear first order DEs (see page 2 of the book).

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