Section 1.2. Some Elementary Matrix Algebra

Note. In this section we give a quick review of some properties of matrices.

Definition. An $m \times n$ *matrix* A is a set of mn numbers a_{ij}, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ which we denote as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}].$$

Definition. If A and B are both $m \times n$ matrices, define the *sum* $A + B = [a_{ij} + b_{ij}]$. If A and B are both $m \times n$, then $A = B$ is $a_{ij} = b_{ij}$ for all i and j. If $\lambda \in \mathbb{R}$, then *scalar Multiplication* of A by λ is $\lambda A = [\lambda a_{ij}]$. The *zero matrix* is the matrix of all zeros (of whatever given size), denoted 0.

Note. Some properties of matrix addition are as follows. For A, B, C and 0 $m \times n$ matrices we have:

1. $A + B = B + A$

2. $A + (B + C) = (A + B) + C$

3. $A - A = 0$

4. $A + 0 = 0 + A = A$

Note. The collection of all $m \times n$ matrices forms an abelian group under addition.

Definition. If A is an $m \times p$ matrix and B is a $p \times n$ matrix, define the *product*

$$AB = [c_{ij}] = \left[\sum_{k=1}^{p} a_{ik} b_{kj} \right]$$

for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

Note. Matrix multiplication is not necessarily commutative.

Definition. Let $A = [a_{ij}]$ be an $m \times n$ matrix. The $n \times m$ matrix $B = [b_{ij}]$ where $b_{ij} = a_{ji}$ is the *transpose* of A, denoted A^T.

Theorem 1.2.1. Let $\alpha \in \mathbb{R}$ and suppose the products below are defined. Then

1. $A(BC) = (AB)C$

2. $\alpha(AB) = (\alpha A)B = A(\alpha B)$

3. $(A + B)C = AC + BC$

4. $C(A + B) = CA + CB$

5. $(AB)^T = B^T A^T$

Definition. An $m \times 1$ matrix is an *m-dimensional* (column) *vector*. The $p \times p$ matrix A such that

$$a_{ij} = \delta(i,j) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

is the *identity matrix*, denoted $\mathcal{I} = \mathcal{I}_p$.

Note. If A is $m \times n$ then $A = AI_n = I_m A$.

Definition. The *determinant* of a 1×1 matrix A is $\det(A) = a_{11}$. The *minor* of an element a_{ij} of a matrix A is the matrix obtained by deleting the ith row and the jth column of A and is denoted M_{ij}. For A an $n \times n$ matrix, define the *determinant* of A recursively as:

$$A_{ij} = (-1)^{i+j}\det(M_{ij}) \quad \text{(these are called } \textit{cofactors}\text{)}$$

$$\det(A) = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{n1}A_{n1}.$$

Theorem 1.2.2. For $n \times n$ matrix A,

$$\det(A) = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij}.$$

Note. Theorem 1.2.2 implies that determinants can be calculated by expanding about any row or column.

Theorem 1.2.3. If A and B are $n \times n$ then

$$\det(AB) = \det(A)\det(B).$$

Definition. A square matrix is *singular* if $\det(A) = 0$ and *nonsingular* if $\det(A) \neq 0$.

Definition. A matrix B is the *inverse* of square matrix A if $AB = \mathcal{I}$. We usually denote B as A^{-1}.

Theorem 1.2.4(a). If A^{-1} exists then $\det(A) \neq 0$.

Theorem 1.2.4(b). If $\det(A) \neq 0$ then A^{-1} exists.

Note. The text constructs A^{-1} as

$$A^{-1} = \frac{1}{\det(A)}[A_{ij}]^T.$$

Example. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\det(A) = ad - bc$ and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note. On page 9 of the text it is shown that A^{-1} is unique and $AA^{-1} = A^{-1}A$.

Theorem 1.2.5. The system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n \end{aligned}$$

has a unique solution if and only if $A = [a_{ij}]$ is nonsingular.

Note. As a matrix equation, the system can be written $A\vec{x} = \vec{c}$ and if A is nonsingular then $\vec{x} = A^{-1}\vec{c}$.

Definition. A set of k n-dimensional vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ is *linearly dependent* if there exists constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\vec{x} - 1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}.$$

A set of vectors which are not linearly dependent is *linearly independent.*

Theorem 1.2.6. A matrix is nonsingular if and only if its columns are linearly independent.

Note. We can also consider a matrix of functions $A(t) = [a_{ij}(t)]$.

Definition. $A(t)$ is

1. continuous at t_0 if each $a_{ij}(t)$ is continuous at t_0,

2. differentiable at t_0 if each $a_{ij}(t)$ is differentiable at t_0, and

3. integrable over $[a, b]$ is each $a_{ij}(t)$ is integrable over $[a, b]$.

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