Section 1.3. The Structure of Solutions of Homogeneous Linear Systems

Note. In this section we introduce matrix notation to deal with systems of linear equations.

Note. The system of n linear first order DEs

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + e_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + e_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + e_n(t) \end{aligned}$$

can be written as $\vec{x}\,' = A(t)\vec{x} + \vec{e}(t)$.

Theorem 1.3.1. Let $A(t)$ be a continuous $n \times n$ matrix defined on interval I and let $\vec{e}(t)$ be a continuous n-dimensional vector defined on I. For every constant n-vector $\vec{x}_0$ and every $t_0 \in I$ there exists a unique differentiable vector $\varphi(t)$ defined on I such that $\vec{\varphi}\,'(t) = A(t)\vec{\varphi}(t) + \vec{e}(t)$ for all $t \in I$ and $\vec{\varphi}(t_0) = \vec{x}_0$.

Note/Definition. The condition "$\vec{\varphi}(t_0) = \vec{x}_0$" is an example of an *initial value.* The DE in Theorem 1.3.1 along with the initial value form an *initial value problem* (or "IVP"). In the even that $\vec{e}(t) = \vec{0}$, then DE $\vec{\varphi}\,'(t) = A(t)\vec{\varphi}(t)$ is said to be *homogeneous.*

Note. We shall see a generalization of Theorem 1.3.1 in Chapter 3 (see Theorem 3.5.1). Such a theorem is called an "existence and uniqueness" theorem.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of functions. A function $T : \mathcal{A} \to \mathcal{B}$ is an *operator*. A is the *domain* of T and $\{y \mid y = Tx \text{ for some } x \in \mathcal{A}\}$ is the *range* of T. If for all $x, y \in \mathcal{A}$ and for all $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

then T is a *linear operator*.

Example. $T = d/dx$ is a linear operator with domain of all differentiable functions (on $\mathbb{R}$, say).

Note. If $A(t) = A$ is an $n \times n$ matrix of continuous functions, then the operator $L[\vec{x}] = \vec{x}\,' - A\vec{x}$ maps the set of n-vectors of continuously differentiable function into the set of n-vectors of continuous functions. Notice that if $\vec{\varphi}$ is a solution of $\vec{x}\,' = A\vec{x} + \vec{e}(t)$ then $L[\vec{\varphi}] = \vec{e}(t)$.

Theorem 1.3.2. If $A(t)$ is an $n \times n$ matrix of continuous functions on an interval I, then $L[\vec{x}] = \vec{x}\,' - A\vec{x}$ is a linear operator.

Theorem 1.3.3. Let L and A be as in Theorem 1.3.2. If $\vec{x}_1$ and $\vec{x}_2$ are solutions of $\vec{x}\,' = A\vec{x}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution.

Definition. Let L be as in Theorem 1.3.2. If $\vec{x}_i(t)$ for $i = 1, 2, \ldots, n$ is a collection of n linearly independent solutions of $L[\vec{x}] = \vec{0}$ then the matrix Φ whose columns are $\vec{x}_i$, $\Phi = [\vec{x}_i]$, is the *fundamental matrix* for the DE $\vec{x}' - A\vec{x} = \vec{0}$.

Note. If $\vec{c}$ is a constant vector, then $\Phi\vec{c}$ is a solution of $L[\vec{x}] = \vec{0}$:

$$L[\Phi\vec{c}] = \Phi'\vec{c} - A\Phi\vec{c} = (\Phi' - A\Phi)\vec{c} = 0\vec{c} = 0.$$

Theorem 1.3.4. Let L be as in Theorem 1.3.2. If Φ is the fundamental matrix for $L[\vec{x}] = \vec{0}$ on an interval I where $A(t)$ is continuous, then every solution of $L[\vec{x}] = \vec{0}$ can be written as $\phi\vec{c}$ for some constant vector $\vec{c}$.

Note. All of the above is based on the existence of Φ.

Theorem 1.3.6. If $A(t)$ is continuous, there exists a fundamental matrix for $L[\vec{x}] = \vec{0}$ where L is as in Theorem 1.3.2.

Note. We still don't know how to find Φ.

Revised: 4/14/2019