Section 1.4. Matrix Analysis and Matrix Exponentiation

Note. In this section we introduce a norm on square matrices and define exponentiation of square matrices.

Definition. If A is an $R \times R$ matrix, define the *norm* of A as $\|A\| = \sum_{i,j} |a_{ij}|$. If $\vec{A}$ is a vector $(a_1, a_2, \ldots, a_R)^T$, define the *norm* as $\|\vec{A}\| = \sum_i |a_i|$.

Theorem 1.4.A. Let A be an $R \times R$ matrix. Then:

1. $\|A\| \geq 0$ for $A \neq 0$, and $\|0\| = 0$.

2. $\|cA\| = |c|\|A\|$ for scalar c.

3. $\|A + B\| \leq \|A\| + \|B\|$.

4. $\|AB\| \leq \|A\|\|B\|$.

5. $\|A\vec{x}\| \leq \|A\|\|\vec{x}\|$ for $\vec{x}$ an R-vector.

Definition. A sequence of $R \times R$ matrices, A_n, is *Cauchy* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have $\|A_n - A_m\| < \varepsilon$.

Theorem 1.4.1. Every Cauchy sequence of matrices (with real entries) A_n has a limit.

Definition. Given a sequence of matrices A_n, define the *nth partial sum* $S_n = A_1 + A_2 + \cdots + A_n$. If the sequence S_n has limit S, then the series $\sum_{i=1}^{\infty} A_i$ is said to *converge* to S. If the sequence S_n does not have a limit, the series $\sum_{i=1}^{\infty} A_i$ is said to *diverge*.

Note. Recall that for $x \in \mathbb{R}$ we have $s^x = \sum_{n=0}^{\infty} x^n/n!$.

Theorem 1.4.2. The series $I + \sum_{n=1}^{\infty} A^n.n!$ converges for all square matrices A.

Definition. For A an $R \times R$ matrix, define $e^A = \sum_{k=0}^{\infty} A^k/n!$ and for $t \in \mathbb{R}$ define $s^{At} = \sum_{k=1}^{\infty} A^k t^k/k!$.

Note. Notice that e^{At} is continuous, differentiable, and integrable with respect to t. Of course

$$\frac{d}{dt}[e^{At}] = Ae^{At} = e^{At}A.$$

Definition. $R \times R$ matrices A and B are *similar* if there exists a nonsingular matrix T such that $T^{-1}AT = B$.

Note. If $B = T^{-1}AT$ then $B^n = T^{-1}A^nT$ and $e^B = T^{-1}e^AT$.

Theorem 1.4.4. For any square matrix M, $\det(e^M) \neq 0$.