Section 1.4. Matrix Analysis and Matrix Exponentiation

Note. In this section we introduce a norm on square matrices and define exponentiation of square matrices.

Definition. If $A$ is an $R \times R$ matrix, define the norm of $A$ as $\|A\| = \sum_{i,j} |a_{ij}|$. If $\vec{A}$ is a vector $(a_1, a_2, \ldots, a_R)^T$, define the norm as $\|\vec{A}\| = \sum_i |a_i|$.

Theorem 1.4.A. Let $A$ be an $R \times R$ matrix. Then:

1. $\|A\| \geq 0$ for $A \neq 0$, and $\|0\| = 0$.
2. $\|cA\| = |c| \|A\|$ for scalar $c$.
3. $\|A + B\| \leq \|A\| + \|B\|$.
4. $\|AB\| \leq \|A\| \|B\|$.
5. $\|A\vec{x}\| \leq \|A\| \|\vec{x}\|$ for $\vec{x}$ an $R$-vector.

Definition. A sequence of $R \times R$ matrices, $A_n$, is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have $\|A_n - A_m\| < \varepsilon$.

Theorem 1.4.1. Every Cauchy sequence of matrices (with real entries) $A_n$ has a limit.
Definition. Given a sequence of matrices $A_n$, define the \textit{nth partial sum} $S_n = A_1 + A_2 + \cdots + A_n$. If the sequence $S_n$ has limit $S$, then the series $\sum_{i=1}^{\infty} A_i$ is said to \textit{converge} to $S$. If the sequence $S_n$ does not have a limit, the series $\sum_{i=1}^{\infty} A_i$ is said to \textit{diverge}.

Note. Recall that for $x \in \mathbb{R}$ we have $s^x = \sum_{n=0}^{\infty} x^n/n!$.

Theorem 1.4.2. The series $I + \sum_{n=1}^{\infty} A^n/n!$ converges for all square matrices $A$.

Definition. For $A$ an $R \times R$ matrix, define $e^A = \sum_{k=0}^{\infty} A^k/n!$ and for $t \in \mathbb{R}$ define $s^{At} = \sum_{k=1}^{\infty} A^k t^k/k!$.

Note. Notice that $e^{At}$ is continuous, differentiable, and integrable with respect to $t$. Of course
\[
\frac{d}{dt}[e^{At}] = Ae^{At} = e^{At}A.
\]

Definition. $R \times R$ matrices $A$ and $B$ are \textit{similar} if there exists a nonsingular matrix $T$ such that $T^{-1}AT = B$.

Note. If $B = T^{-1}AT$ then $B^n = T^{-1}A^nT$ and $e^B = T^{-1}e^{AT}$.

Theorem 1.4.4. For any square matrix $M$, $\det(e^M) \neq 0$.  

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