Section 1.5. The Constant Coefficient Case: Real and Distinct Eigenvalues

Note. In this section we consider diagonalizing matrices which have real and distinct eigenvalues and solve systems of linear equations determined by such (constant) matrices.

Note. If λ is a complex eigenvalue of A and $\vec{c}$ is an eigenvector corresponding to λ then $e^{\lambda t}\vec{c}$ is still a solution of $\vec{y}\,' = A\vec{y}$.

Theorem 1.5.1. Let A be a constant matrix. A fundamental matrix Φ for $y' = Ay$ is $\Phi = e^{At}$.

Definition. Two square matrices A and B are *similar* if there exists invertible T such that $B = TAT^{-1}$. A matrix which is similar to a diagonal matrix is said to be *diagonalizable*.

Note. As we have already seen, it is easy to raise a diagonalizable matrix to powers:

$$\begin{aligned} B^n = (TAT^{-1})^n &= \underbrace{(TAT^{-1})(TAT^{-1})\cdots(TAT^{-1})}_{n \text{ times}} \\ &= TA(T^{-1}T)A(T^{-1}T)A\cdots A(T^{-1}T)AT^{-1} = TA^nT^{-1}. \end{aligned}$$

Definition. For a square matrix A, any complex or real number λ such that $\det(A - \lambda\mathcal{I}) = 0$ is an *eigenvalue* of A. A vector $\vec{c} \neq \vec{0}$ such that $(A - \lambda\mathcal{I})\vec{c} = \vec{0}$ is an *eigenvector* of A for the eigenvalue λ.

Note. If λ is an eigenvalue of A, then $A - \lambda\mathcal{I}$ is not invertible by Theorem 1.2.4. Also, $(A - \lambda\mathcal{I})\vec{x} - \vec{0}$ must not have a unique solution by Theorem 1.2.5. $\vec{x} = \vec{0}$ is of course one solution, so there must be other solutions to this equation (and therefore there exist eigenvectors for any eigenvalue).

Theorem 1.5.2. If $B = TAT^{-1}$, then A and B have the same eigenvalues.

Definition. If A is $n \times n$, then the n-degree polynomial of λ, $\det(A - \lambda\mathcal{I})$, is the *characteristic polynomial* of A.

Note. Since $\det(A - \lambda\mathcal{I})$ is an n-degree polynomial, A has n eigenvalues (counting multiplicity; this follows from the Fundamental Theorem of Arithmetic).

Example. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. HINT: The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -3$, with corresponding sets of eigenvectors of $\vec{c}_1 \in \left\{ r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| r \in \mathbb{R}, r \neq 0 \right\}$ and $\vec{c}_1 \in \left\{ s \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| s \in \mathbb{R}, s \neq 0 \right\}$.

Note. If λ is an eigenvalue of A and $\vec{c}$ is a corresponding eigenvalue, then $(A - \lambda\mathcal{I})\vec{c} = \vec{0}$, or $A\vec{c} = \lambda\vec{c}$.

Theorem 1.5.3. If A is a constant matrix, λ is an eigenvalue of A and $\vec{c}$ a corresponding eigenvector, then $\vec{y} = e^{\lambda t}\vec{c}$ is a solution of $\vec{y}\,' = A\vec{y}$.

Theorem 1.5.A. (From Linear Algebra.) If A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$, then the set $\{\vec{c} - 1, \vec{c}_2, \ldots, \vec{c}_n\}$ is linearly independent and if $T = [\vec{c}_i]$ then $A = TDT^{-1}$ where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is diagonal.

Theorem 1.5.4. Let A be a constant $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$. Then a fundamental matrix for $\vec{y}\,' = A\vec{y}$ is $\Phi(t) = [e^{\lambda_1 t}\vec{c}_1 \ \ e^{\lambda_2 t}\vec{c}_2 \ \ \cdots \ \ e^{\lambda_n t}\vec{c}_n]$.

Example. Solve $\vec{y}\,' = A\vec{y}$ for $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$. (Notice that the eigenvalues are negative and general solutions approach $\vec{0}$ as $t \to \infty$.)

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