Section 1.6. The Constant Coefficient Case: Complex and Distinct Eigenvalues

Note. In this section we consider diagonalizing matrices which have complex and distinct eigenvalues and solve systems of linear equations determined by such (constant) matrices.

Note. If λ is a complex eigenvalue of A and $\vec{c}$ is an eigenvector corresponding to λ then $e^{\lambda t}\vec{c}$ is still a solution of $\vec{y}' = A\vec{y}$.

Note. The following takes advantage of the fact that we are dealing with *linear* systems of equations.

Theorem 1.6.1. If $\vec{\varphi}(t)$ is a solution of $\vec{x}' = A\vec{x}$ where A is a constant matrix (with real entries) then $\mathrm{Re}(\vec{\varphi}(t))$ and $\mathrm{Im}(\vec{\varphi}(t))$ are also solutions.

Note. Euler's formula states that $e^{i\theta} = \cos\theta_i \sin\theta$. Then if $\lambda = \alpha + i\beta$ is an eigenvalue of A and $\vec{c} = \vec{a} + i\vec{b}$ then

$$\mathrm{Re}(e^{(\alpha+i\beta)t}(\vec{a}+i\vec{b})) = e^{\alpha t}(\vec{a}\cos\beta t - \vec{b}\sin\beta t)$$

$$\text{and } \mathrm{Im}(e^{(\alpha+i\beta)t}(\vec{a}+i\vec{b})) = e^{\alpha t}(\vec{a}\sin\beta t - \vec{b}\cos\beta t).$$

These two vectors are linear independent.

Note. Since for us the characteristic polynomial for a matrix A has all real coefficients, if λ is an eigenvalue of A so is $\overline{\lambda}$. Notice that $\overline{\lambda} = \alpha - i\beta$ yields "the same" (up to a constant multiple, at least) eigenvectors as does λ (so we are getting one eigenvector per eigenvalue, up to linear independence).

Example. Solve $\vec{x}\,' = A\vec{x}$ for $A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$.

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