Section 1.8. General Linear Systems

Note. In this section we find solutions to nonhomogeneous linear systems of equations.

Theorem 1.8.1. If $\vec{x}(t)$ is a solution of

$$\vec{y}\,' - A\vec{y} = \vec{e}(t) \qquad (8.2)$$

($\vec{e}(t)$ is called a *forcing term*), then any solution $\vec{\Psi}(t)$ of (8.2) can be written as

$$\vec{\Psi}(t) = \Phi(t)\vec{c} + \vec{x}(t)$$

where Φ is a fundamental matrix for

$$\vec{y}\,' - A\vec{y} = \vec{0} \qquad (8.3)$$

and $\vec{c}$ is a constant.

Theorem 1.8.2.

$$\vec{x}(t) - \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)\vec{e}(\tau)\,d\tau$$

is a solution of $\vec{y}\,' = \vec{e}(t)$.

Note. Theorem 1.8.2 can be modified to give solutions of IVPs, as follows.

Theorem 1.8.3. Variation of Constants.

Let $\Phi(t)$ be a fundamental matrix for $\vec{x}' = A(t)\vec{x}$. Then the solution of

$$\begin{cases} \vec{y}' = A(t)\vec{y} + \vec{e}(t) \\ \vec{y}(t_0) = \vec{\eta} \end{cases}$$

is

$$\vec{y}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{\eta} + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\vec{e}(s)\,ds.$$

Example. Page 68 Number 1(a). Solve $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}$.

Partial Solution. We have $\det(A - \lambda\mathcal{I}) = (1-\lambda)(2-\lambda)$ and so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. We find that we have eigenvectors $\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{c}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So $\Phi(t) = \begin{bmatrix} e^t & 2e^{2t} \\ 0 & e^{2t} \end{bmatrix}$ and $\Phi^{-1}(t) = e^{-3t} \begin{bmatrix} e^{2t} & -2e^{2t} \\ 0 & e^t \end{bmatrix} = \begin{bmatrix} e^{-1} & -2e^{-t} \\ 0 & e^{-2t} \end{bmatrix}$. So we have

$$\vec{x}(t) = \Phi(t) \int_0^t \Phi^{-1}(\tau)\vec{e}(\tau)\,d\tau = \begin{bmatrix} e^t & 2e^{2t} \\ 0 & e^{2t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-\tau} & -2e^{-\tau} \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} d\tau.$$

We can now find the general solution using Theorem 1.8.3.

Revised: 4/14/2019