Section 2.2. Phase Plane

Note. In this section we define the phase plane and classify equilibrium points as stable or unstable.

Note. In the previous section we considered the IVP:

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad (1.2)$$

$$x(t_0) = \alpha, y(t_0) = \beta$$

where $\prime = d/dt$. Solutions to (1.2) (guaranteed to exist by Theorem 2.1.1) can be viewed as triples $(t, x(t), y(t)$ in $\mathbb{R}^3$ or as curves $(x(t), y(t))$ in $\mathbb{R}^2$ which are given parametrically.

Definition. The *phase plane* associated with (1.2) is the (x,y) plane and the curve $(x(t), y(t))$ is a *trajectory* of (1.2) which passes through (α, β) when $t = 0$.

Lemma 2.2.1. If $(\varphi_1(t), \varphi_2(t))$ is a solution of (1.2), then so is $(\varphi_1(t-\tau), \varphi_2(t-\tau))$ for any $\tau \in \mathbb{R}$.

Theorem 2.2.2. Let f and g be continuously differentiable. Through each point (x_0, y_0) there is a unique trajectory of $\begin{matrix} x' = f(x,y) \\ y' = g(x,y). \end{matrix}$

Note. The two dimensional system $\begin{matrix} x' = f(x,y) \\ y' = g(x,y). \end{matrix}$ is equivalent to (by the Chain Rule):

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{dx/dt} = \frac{g(x,y)}{f(x,y)}.$$

Definition. A two dimensional system which can be written $\begin{matrix} x' = f(x,y) \\ y' = g(x,y). \end{matrix}$ (where $' = d/dt$) is an *autonomous system* (that is, the derivatives of x and y do not depend explicitly on t. In other words, the system changes according to the state it is in and is independent of "time").

Definition. For the system (1.2), (x_0, y_0) is a *critical point* if $f(x_0, y_0) = g(x_0, y_0) = 0$. This is also called an *equilibrium solution* or a *stationary solution.*

Theorem 2.2.A. The only trajectory which passes through a critical point (x_0, y_0) is the constant solution $x = x_0, y = y_0$.

Example. Page 107 Number 6d.

Definition. A critical point (x_0, y_0) of (1.2) is *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $(x(t), y(t))$ is a solution of (1.2) and if $\|(x(t_0), y(t_0)) - (x_0, y_0)\| < \delta$ then $\|(x(t), y(t)) - (x_0, y_0)\| < \varepsilon$ for all $t \geq t_0$. A critical point that is not stable is *unstable.*

Definition. A stable critical point is *asymptotically stable* if there exists $\eta > 0$ such that if $\|(x(t_0), y(t_0) - (x_0, y_0)\| < \eta$ then $\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$.

Note. If we represent $\begin{array}{l} x' = ax + by \\ y' = cx + dy \end{array}$ in polar coordinates, then we find that $x = r\cos\theta$, $y = r\sin\theta$ implies

$$r' = r(a\cos^2\theta + d\sin^2\theta + (b+c)\sin\theta\cos\theta)$$

$$\theta' = c\cos^2\theta - b\sin^2\theta + (d-a)\sin\theta\cos\theta.$$

Example. Page 106 Number 2.

Note. If we let $\begin{array}{l} \varphi_1(r) = r(t)\cos\theta(t) \\ \varphi_2(t) = r(t)\sin\theta(t) \end{array}$ be a solution of $\begin{array}{l} x' = ax + by \\ y' = cx + dy \end{array}$ then

$$\varphi_1(t) = ar(t)\cos\theta(t) + br(t)\sin\theta(t)$$

$$\varphi_2(t) = cr(t)\cos\theta(t) + dr(t)\sin\theta(t)$$

or

$$r'(t) = r(a\cos^2\theta + d\sin^2\theta + (b+c)\sin\theta\cos\theta)$$

$$\theta'(t) = c\cos\cos^2\theta - b\sin^2\theta + (d-a)\sin\theta\cos\theta.$$

Example. Page 106 Number 2.