Section 2.2. Phase Plane

Note. In this section we define the phase plane and classify equilibrium points as stable or unstable.

Note. In the previous section we considered the IVP:

\[
\begin{align*}
  x' &= f(x, y) \\
  y' &= g(x, y)
\end{align*}
\] (1.2)

\[x(t_0) = \alpha, y(t_0) = \beta\]

where \( t = \frac{d}{dt} \). Solutions to (1.2) (guaranteed to exist by Theorem 2.1.1) can be viewed as triples \((t, x(t), y(t))\) in \( \mathbb{R}^3 \) or as curves \((x(t), y(t))\) in \( \mathbb{R}^2 \) which are given parametrically.

Definition. The phase plane associated with (1.2) is the \((x, y)\) plane and the curve \((x(t), y(t))\) is a trajectory of (1.2) which passes through \((\alpha, \beta)\) when \(t = 0\).

Lemma 2.2.1. If \((\varphi_1(t), \varphi_2(t))\) is a solution of (1.2), then so is \((\varphi_1(t-\tau), \varphi_2(t-\tau))\) for any \(\tau \in \mathbb{R}\).

Theorem 2.2.2. Let \(f\) and \(g\) be continuously differentiable. Through each point \((x_0, y_0)\) there is a unique trajectory of

\[
\begin{align*}
  x' &= f(x, y) \\
  y' &= g(x, y)
\end{align*}
\]
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Note. The two dimensional system \( x' = f(x, y) \) is equivalent to (by the Chain Rule):
\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{1}{dx/dt} = \frac{g(x, y)}{f(x, y)}.
\]

Definition. A two dimensional system which can be written \( x' = f(x, y) \) (where \( y' = g(x, y) \).
\( t = d/dt \)) is an autonomous system (that is, the derivatives of \( x \) and \( y \) do not depend explicitly on \( t \). In other words, the system changes according to the state it is in and is independent of “time”).

Definition. For the system (1.2), \((x_0, y_0)\) is a critical point if \( f(x_0, y_0) = g(x_0, y_0) = 0 \). This is also called an equilibrium solution or a stationary solution.

Theorem 2.2.A. The only trajectory which passes through a critical point \((x_0, y_0)\) is the constant solution \( x = x_0, y = y_0 \).

Example. Page 107 Number 6d.

Definition. A critical point \((x_0, y_0)\) of (1.2) is stable if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \((x(t), y(t))\) is a solution of (1.2) and if \( \|(x(t_0), y(t_0)) - (x_0, y_0)\| < \delta \) then \( \|(x(t), y(t)) - (x_0, y_0)\| < \varepsilon \) for all \( t \geq t_0 \). A critical point that is not stable is unstable.
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**Definition.** A stable critical point is *asymptotically stable* if there exists \( \eta > 0 \) such that if \( \| (x(t_0), y(t_0)) - (x_0, y_0) \| < \eta \) then \( \lim_{t \to \infty} (x(t), y(t)) = (x_0, y_0) \).

**Note.** If we represent \( x' = ax + by \) in polar coordinates, then we find that \( x = r \cos \theta, \ y = r \sin \theta \) implies

\[
\begin{align*}
r' &= r(a \cos^2 \theta + d \sin^2 \theta + (b + c) \sin \theta \cos \theta) \\
\theta' &= c \cos^2 \theta - b \sin^2 \theta + (d - a) \sin \theta \cos \theta.
\end{align*}
\]

**Example.** Page 106 Number 2.

**Note.** If we let \( \varphi_1(r) = r(t) \cos \theta(t) \) be a solution of \( x' = ax + by \) then \( \varphi_2(t) = r(t) \sin \theta(t) \) be a solution of \( y' = cx + dy \) then

\[
\begin{align*}
\varphi_1(t) &= ar(t) \cos \theta(t) + br(t) \sin \theta(t) \\
\varphi_2(t) &= cr(t) \cos \theta(t) + dr(t) \sin \theta(t)
\end{align*}
\]

or

\[
\begin{align*}
r'(t) &= r(a \cos^2 \theta + d \sin^2 \theta + (b + c) \sin \theta \cos \theta) \\
\theta'(t) &= c \cos^2 \theta - b \sin^2 \theta + (d - a) \sin \theta \cos \theta.
\end{align*}
\]

**Example.** Page 106 Number 2.