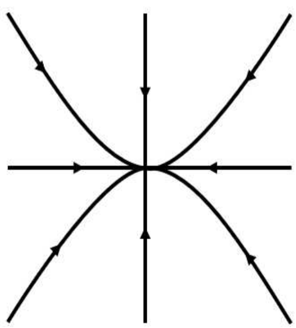
Section 2.3. Critical Points of Some Special Linear Systems

Note. In this section we consider $\vec{x}' = A\vec{x}$ where A is a 2×2 constant matrix. We go through several cases based on the eigenvalues of A.

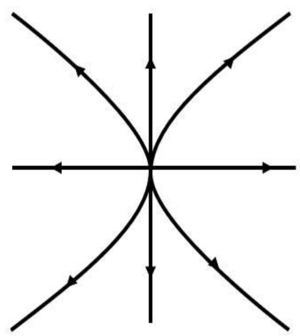
Case 1. Suppose the eigenvalues of A are real, distinct, and of the same sign. Then solutions are of the form

$$x(t) = x_0 e^{\lambda t}$$
$$y(t) = y_0 e^{\mu t}$$

where $\lambda \neq \mu$. In the phase plane the origin is called a *node* and we have:



$\mu < \lambda < 0$

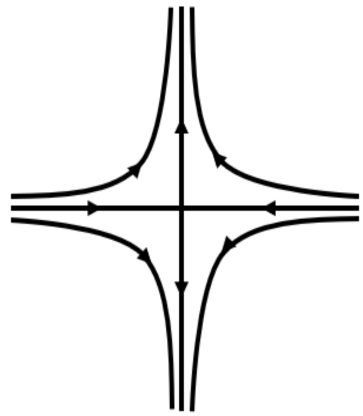


$0 < \mu < \lambda$

Case 2. Suppose the eigenvalues of A are real and of opposite signs, say $\lambda < 0 < \mu$. Then solutions are of the form

$$x(t) = x_0 e^{\lambda t}$$
$$y(t) = y_0 e^{\mu t}.$$

In the phase plane the origin is called a *saddle point* and we have:

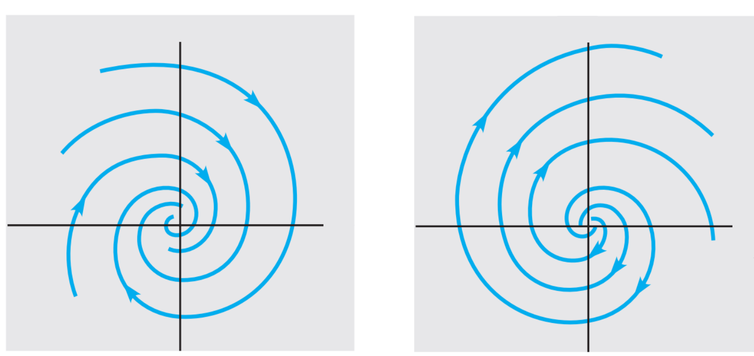


Case 3. Suppose the eigenvalues of A are complex conjugates with nonzero real parts. Consider $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ where $\alpha\beta \neq 0$, which has $\alpha \pm i\beta$ as eigenvalues. We have $\begin{aligned} x' &= \alpha x + \beta y \\ y' &= -\beta x + \alpha y \end{aligned}$ or, in polar coordinates, $\begin{aligned} r' &= \alpha r \\ \theta' &= -\beta. \end{aligned}$ Then the solution is of the form

$$r = r_0 e^{\alpha t}$$

$$\theta = \theta_0 = \beta t.$$

In the phase plane we have for $\alpha < 0$ and $\beta > 0$ that the origin is asymptotically stable (and is called a *spiral point*; left) and for $\alpha > 0$ and $\beta > 0$ the origin is unstable (right):



This image is from *Elementary Differential Equations and Boundary Value Prob-*

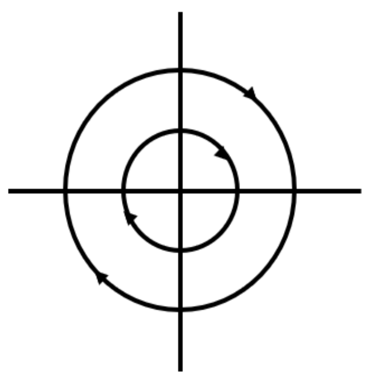
lems, 5th Edition by Richard Diprima and William Boyce, John Wiley & Sons (1991).

Example. Page 115 Exercise 8(b). Solve $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Case 4. Suppose the eigenvalues of A are purely imaginary. Consider $A = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$ or, in polar coordinates, $\begin{matrix} r' = 0 \\ \theta' = -\beta \end{matrix}$ we get

$$\begin{aligned} r &= r_0 \\ \theta &= -\beta t + \theta_0. \end{aligned}$$

In the phase plane the origin is stable (and called a *center*):

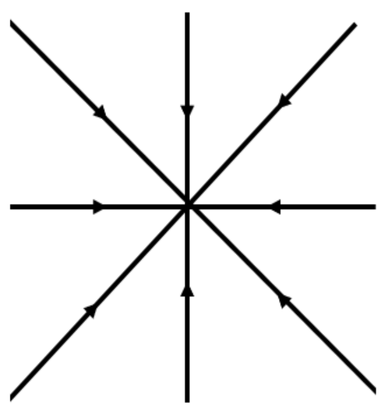


Case 5. Suppose the eigenvalues of A are the same. Consider $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ or

$\begin{aligned} x' &= \lambda x \\ y' &= \lambda y \end{aligned}$ and we get

$$\begin{aligned} x &= x_0 e^{\lambda t} \\ y &= y_0 e^{\lambda t}. \end{aligned}$$

We see, in polar coordinates, $r^x = x^2 + y^2 = e^{2\lambda t}(x_0^2 + y_0^2)$ and $\tan\theta = y/x = y_0/x_0$ (a constant). In the phase plane the origin is called a *degenerate node* (trajectories approach the origin at the same rate from any direction, in contrast to Case 1):

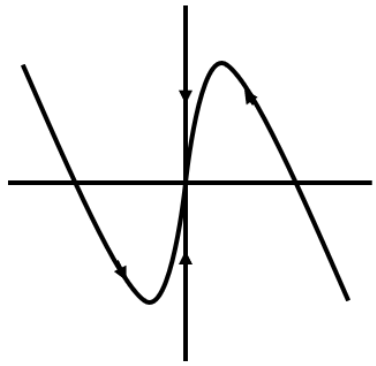


Also consider $A = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ or $\begin{aligned} x' &= \lambda x \\ y' &= x + \lambda y \end{aligned}$ and we get

$$\begin{aligned} x &= x_0 e^{\lambda t} \\ y &= y_0 e^{\lambda t} + x_0 t e^{\lambda t}. \end{aligned}$$

If $x_0 \neq 0$, then in polar coordinates $\tan\theta = (y_0 + t x_0)/x_0$ and $\lim_{t\to\infty} \theta = \pm\pi/2$.

In the phase plane:



Examples. Page 115 Numbers 9(b) and 10(b).

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