Section 2.4. Critical Points of General 2-D Linear Systems

Note. In this section we diagonalize 2×2 matrices and solve 2-D systems.

Note. Let matrix B^{-1} be 2×2, invertible, and define $\begin{bmatrix} z \\ w \end{bmatrix} = B^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$ for given $\begin{bmatrix} x \\ y \end{bmatrix}$. If $\begin{bmatrix} x \\ y \end{bmatrix}$ is a solution to $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = B^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = B^{-1}A \begin{bmatrix} x \\ y \end{bmatrix} = B^{-1}AB \begin{bmatrix} z \\ w \end{bmatrix}$$

and we see $\begin{bmatrix} z \\ w \end{bmatrix}$ is a solution of $\begin{bmatrix} z' \\ w' \end{bmatrix} = T \begin{bmatrix} z \\ w \end{bmatrix}$ if $T = B^{-1}AB$. Notice that T and A are similar and therefore have the same eigenvalues (matrix T will be one of the six types of matrices discussed in Section 2.3).

Example. (Exercise 2.4.4.) If $D = T^{-1}AT$ is a diagonal matrix, then the columns of T are eigenvectors of T are eigenvectors of A and the nonzero entries of D are eigenvalues of A.

Example. Consider $A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$. Find matrices B and B^{-1} such that $D = B^{-1}AB$ is diagonal. SOLUTION: We have $B = \begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix}$, and $B^{-1} = -\frac{1}{9}\begin{bmatrix} -1 & -1 \\ -5 & 4 \end{bmatrix}$.

Example. Solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

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