Section 2.5. Behavior of Nonlinear 2-D Systems Near a Critical Point

Note. In this section we linearize a nonlinear system and analyze the stability of critical points.

Note. We assume for the remainder of Chapter 2 that

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad (5.1)$$

where f and g have continuous partial derivatives.

Note. Recall **Taylor's Theorem**: Let f be such that $f, f', \ldots, f^{(n)}$ are continuous on an open interval containing $[a,b]$ and suppose $f^{(n+1)}$ exists on (a,b). Then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Note. The DE (5.1) becomes

$$\begin{cases} x' = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \varepsilon_1(x-x_0, y-y_0) \\ y' = g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) + \varepsilon_2(e-x_0, y-y_0) \end{cases} \qquad (5.2)$$

where ε_1 and ε_2 are functions. Letting $z_1 = x - x_0$, $z_2 = y - y_0$ we get

$$\begin{cases} z_1' = az_1 + bz_2 + \varepsilon_1(z_1,z_2) \\ z_2' = cz_1 + dz_2 + \varepsilon_2(z_1,z_2) \end{cases} \qquad (5.3)$$

where $\varepsilon_1(z_1,z_2)$ is continuously differentiable, $\varepsilon_2(0,0) = 0$, and $\dfrac{\partial \varepsilon_i}{\partial z_j}(0,0) = 0$ (i.e., ε_2 contains no linear part).

Note. We find that the behavior of critical point of $(5,3)$ and of

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad (1.1)$$

are "intimately related."

Theorem 1.5.1. If the origin is an asymptotically stable critical point for (1.1), then it is an asymptotically critical point of (5.3). If the origin is unstable for (1.1), then it is unstable for (5.3).

Note. No comment is made in the case of the origin as a stable but not asymptotically stable critical point of (1.1) (i.e., a center).

Definition. The autonomous systems

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad (5.7)$$

and

$$\begin{cases} u' = \alpha(u,v) \\ v' = \beta(u,v) \end{cases} \qquad (5.8)$$

have the same *qualitative structure* in regions G_1 and G_2 if there is one to one and onto $T : G_1 \to G_2$ such that

1. T and T^{-1} are continuous.

2. If two points of G_1 lie on the same trajectory of (5.7), their images under T lie on the same trajectory of (5.8).

3. If two points of G_2 lie on the same trajectory of (5.8), their images under T^{-1} lie on the same trajectory of (5.7).

The mapping T is called a *homeomorphism* of G_1 to G_2.

Theorem 2.5.2. If the real parts of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are nonzero, then (5.3) and (1.1) have the same qualitative structure in a neighborhood of the origin.

Definition. (1.1) is called the *linearization* of (5.3).

Theorem 2.5.3. If the real parts of the eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are positive (respectively, negative) then there is a closed curve about the origin that trajectories of (5.3) cross from inside to outside (respectively, outside to inside) as time increases.

Definition. The critical point (x_0, y_0) of (5.1) is a *spiral point* if, after translating (x_0, y_0) to $(0, 0)$ (as in (5.3)) and changing to polar coordinates (as in (5.6)), it follows that $\lim_{t\to\infty} r(t) = 0$ (or $\lim_{t\to-\infty} r(t) = 0$, respectively) and $\lim_{t\to\infty} \theta(t) = \pm\infty$ (or $\lim_{t\to-\infty} \theta(t) = \pm\infty$, respectively).

Theorem 2.5.4. If the linearization of (5.1) about (x_0, y_0) has the origin as a spiral point, then (x_0, y_0) is a spiral point for (5.3).

Example. Solve
$$\begin{aligned} x' &= y \\ y' &= x - y + x(x - 2y). \end{aligned}$$

Solution. Setting $x' = 0$ and $y' = 0$, we see that $(0,0)$ and $(-1,0)$ are critical points. We have $f_x = 0$, $f_y = 1$, $g_x = 1 + 2x - 2y$, and $g_y = -1 - 2x$. So at $(0,0)$ the linearization is

$$\begin{cases} x' = f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ y' = g_x(0,0)(x-0) + g_y(0,0)(y-0) \end{cases}$$

or

$$\begin{cases} x' = y \\ y' = x - y \end{cases}$$

so that $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. For A, $\lambda^2 - \lambda - 1 = 0$ and so the eigenvalues of A are $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$. So the eigenvalues are real and of opposite signs, so that the origin is a saddle point. At $(-1,0)$ the linearization is

$$\begin{cases} x' = y \\ y' = -x + y \end{cases}$$

so that $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. For A, $\lambda^2 - \lambda + 1 = 0$ and so the eigenvalues of A are $\lambda = \dfrac{1 \pm \sqrt{-3}}{2}$. So the eigenvalues are complex with positive real part and $(-1,0)$ is an unstable critical point.

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