Section 3.2. Preliminaries

Note. We set up a theoretical framework for the study of DEs.

Definition. A continuous function $\varphi(t)$ defined on interval I is a *solution on I of the integral equation*

$$y = y_0 + \int_{t_0}^{t} f(s, y)\, ds$$

if for every $t \in I$, $f(t, \varphi(t))$ is defined and

$$\varphi(t) = y_o + \int_{t_o}^{t} f(x, \varphi(s))\, ds.$$

Theorem 3.2.1. Let $f(t, y)$ be continuous. A function $\varphi(t)$ defined on interval I is a solution to the IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

on I if and only if φ is a continuous solution of the integral equation

$$y = y_0 + \int_{t_0}^{t} f(s, y)\, ds.$$

Definition. A function $f(t, y)$ is *Lipschitz* in variable y in a region $\Omega \subset \mathbb{R}^2$ if there exists $K \in \mathbb{R}$ such that if $(t, y_1), (t, y_2) \in \Omega$ then

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|.$$

Definition. Let M be a set. A function $\rho : M \to \mathbb{R}^2$ is a *metric* on M if

1. $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.

2. $\rho(x, y) = \rho(y, x)$.

3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ for all $x, y, z \in M$.

The pair (M, ρ) is a *metric space*.

Definition. The set of all continuous functions on $[a, b]$ is denoted $C([a, b])$.

Theorem 3.2.A. $C([a, b])$ is a metric space under the metric

$$\rho(f, g) = \max_{y \in [a,b]} |f(t) - g(t)|.$$

Definition. A sequence $\{x_n\} \subset M$ is a *Cauchy sequence* in metric space (M, ρ) if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $\rho(x_n, x_m) < \varepsilon$. A sequence $\{x_n\}$ *converges with limit* $y \in M$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\rho(y, x_n) < \varepsilon$.

Definition. A metric space M is *complete* if every Cauchy sequence converges.

Note. Convergent sequences are Cauchy by the Triangle Inequality.

Theorem 3.2.2. The space $C([a, b])$ is complete.

Definition. A *linear space* (over $\mathbb{R}$) is a set L such that if $f, g \in L$ and $a, b \in \mathbb{R}$ then $af + bg \in L$. A *norm* $\|\cdot\|$ on a linear space is a mapping from L to $\mathbb{R}$ such that

1. $\|f\| = 0$ if and only if $f = 0$,

2. $\|\lambda f\| = |\lambda|\|f\|$ for all $f \in L$ and for all $\lambda \in \mathbb{R}$, and

3. $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in L$.

Note. In a normed linear space, we can define a metric as $d(f, g) = \|f - g\|$.

Definition. A complete normed linear space is called a *Banach space*.

Note. $C([a, b])$ is a Banach space with norm $\|f\| = \max_{t \in [a,b]} |f(t)|$.

Definition. Let $f_0 \in C([a, b])$ and $\alpha \in \mathbb{R}$. Define the set B of all elements f of $C([a, b])$ such that $\rho(f, f_0) \leq \alpha$.

Lemma 3.2.3. The space B is complete.