Section 3.3. The Contraction Mapping Theorem

Note. We give one definition and theorem. The theorem is useful in establishing the existence and uniqueness of solutions to DEs.

Definition. Let $T : M \to M$ where (M, ρ) is a metric space. T is a *contraction* if for all $x, y \in M$ we have $\rho(Tx, Ty) \leq \alpha\rho(x, y)$ where $0 \leq \alpha < 1$.

Example. The function $f(x) = \cos x$ is a contraction on $(0, \pi/2)$.

Theorem 3.3.1. The Contraction Mapping Theorem.
A contraction mapping T defined on a complete metric space has a unique fixed point.

Definition. The set of all continuous functions on $[a, b]$ is denoted $C([a, b])$.

Example. Approximate the fixed point of $f(x) = \cos x$ on $(0, \pi/2)$. As seen in the proof of Theorem 3.3.1, all we have to do is iterate f by starting at any point of $(0, \pi/2)$. You can set a calculator in radians mode, input any value in $(0, \pi/2)$, and repeatedly hit the cosine button and the output will approach the fixed point. The value is approximately 0.73908513.

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