Section 3.4. The Initial Value Problem for One Scalar Differential Equation

Note. We state and prove several theorems concerning the existence of solutions to IVPs.

Theorem 3.4.1. Let $f(x,t)$ be continuous and Lipschitz with Lipschitz constant K on $\Omega = \{(t,y) \mid |t-t_0| \le a, |y-y_0| \le b\}$ and let M be a number such that $|f(t,y)| \le M$ for all $(x,y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

for $|t-t_0| \le \alpha$.

Theorem 3.4.2. Let $f(t,y)$ be continuous and Lipschitz with Lipschitz constant K valid for every t and y (i.e., f is "uniformly Lipschitz"). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

and this solution is valid for all t.

Corollary 3.4.3. If $a(t)$ and $b(t)$ are continuous in $\mathbb{R}$, there exists a unique solution $y(t)$ of

$$y' = a(t)y + b(t)$$
$$y(t_0) = y_0$$

valid for all t.

Lemma 3.4.4. Gronwall's Inequality.

Let $\varphi(t)$ be a nonnegative function where

$$\varphi(t) \leq C + K \int_{t_0}^{t} \varphi(s)\, ds, \;\; t > t_0$$

where C and K are constants, $K \geq 0$ and $C > 0$. Then $\varphi(t) \leq Ce^{K(t-t_0)}$ for $t > t_0$.

Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions.

Define $T : \mathbb{R} \to C([a,b])$ be defined as $Ty_0 = \varphi$ where φ is the solution of

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

for given Lipschitz f with Lipschitz constant K valid for every t and y. Then T is continuous.

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