Section 3.4. The Initial Value Problem for One Scalar Differential Equation

Note. We state and prove several theorems concerning the existence of solutions to IVPs.

**Theorem 3.4.1.** Let $f(x, t)$ be continuous and Lipschitz with Lipschitz constant $K$ on $\Omega = \{(t, y) \mid |t - t_0| \leq a, |y - y_0| \leq b\}$ and let $M$ be a number such that $|f(t, y)| \leq M$ for all $(x, y) \in \Omega$. Choose $0 < \alpha < \min\{1/K, b/M, a\}$. Then there exists a unique solution of

$$\begin{align*}
y' &= f(t, y) \\
y(t_0) &= y_0
\end{align*}$$

for $|t - t_0| \leq \alpha$.

**Theorem 3.4.2.** Let $f(t, y)$ be continuous and Lipschitz with Lipschitz constant $K$ valid for every $t$ and $y$ (i.e., $f$ is “uniformly Lipschitz”). Then for any $t_0, y_0 \in \mathbb{R}$, there is a solution to

$$\begin{align*}
y' &= f(t, y) \\
y(t_0) &= y_0
\end{align*}$$

and this solution is valid for all $t$. 

Corollary 3.4.3. If \( a(t) \) and \( b(t) \) are continuous in \( \mathbb{R} \), there exists a unique solution \( y(t) \) of
\[
\begin{align*}
y' &= a(t)y + b(t) \\
y(t_0) &= y_0
\end{align*}
\]
valid for all \( t \).

Lemma 3.4.4. Gronwall’s Inequality.
Let \( \varphi(t) \) be a nonnegative function where
\[
\varphi(t) \leq C + K \int_{t_0}^{t} \varphi(s) \, ds, \quad t > t_0
\]
where \( C \) and \( K \) are constants, \( K \geq 0 \) and \( C > 0 \). Then \( \varphi(t) \leq C e^{K(t-t_0)} \) for \( t > t_0 \).

Theorem 3.4.5. Continuous Dependence of IVPs on Initial Conditions.
Define \( T : \mathbb{R} \to C([a,b]) \) be defined as \( Ty_0 = \varphi \) where \( \varphi \) is the solution of
\[
\begin{align*}
y' &= f(t,y) \\
y(t_0) &= y_0
\end{align*}
\]
for given Lipschitz \( f \) with Lipschitz constant \( K \) valid for every \( t \) and \( y \). Then \( T \) is continuous.

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