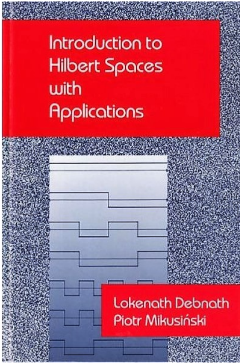
Advanced Differential Equations

Chapter 1. Normed Vector Spaces

Section 1.2. Vector Spaces—Proofs of Theorems



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Let $p > 1$, $q > 1$, and $1/p + 1/q = 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} .$$

Proof. First, by Problem 1.1.5, $x^{1/p} \leq \frac{1}{p}x + \frac{1}{q}$ for $x \in [0,1]$. Let a and b be positive real numbers such that $a^p \leq b^q$. Then $0 \leq a^p/b^q \leq 1$ and so

$$ab^{-q/p} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$$

with $x = a^p/b^q$ in problem 1.1.5. Now

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ implies } \frac{q}{p} + 1 = q \text{ or } \frac{-q}{p} = 1 - q \ldots$$

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Theorem 1.2.1 (continued 1)

Proof (continued). ... and so

$$ab^{1-q} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}.$$

Multiplying by b^q gives

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \qquad (1.2.1)$$

We can show that (1.2.1) also holds when $b^q \leq a^p$. Therefore (1.2.1) holds for any $a, b \geq 0$. If we take

$$a = \frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \text{ and } b = \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}$$

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$$ab = \frac{|x_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}} \frac{|y_j|}{\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$= \frac{1}{p}\frac{|x_j|^p}{\sum_{k=1}^n |x_k|^p} + \frac{1}{q}\frac{|y_j|^q}{\sum_{k=1}^n |y_k|^q}.$$

Summing we have

$$\frac{\sum_{j=1}^n |x_j||y_j|}{\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\left(\sum_{k=1}^n |y_k|^q\right)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Cross multiplying and letting $n \to \infty$ we get the desired result. □

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Theorem 1.2.4. Minkowski's Inequality

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Let $p \geq 1$. For any two sequences of complex numbers $\{x_n\}$ and $\{y_n\}$ we have

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}.$$

Proof. If $p = 1$, the result holds by the Triangle Inequality for absolute value. If $p > 1$, then there exists q such that $1/p + 1/q = 1$ and by Hölder's Inequality we have

$$\sum_{k=1}^{\infty} |x_n + y_n|^p = \sum_{n=1}^{\infty} |x_n + y_n||x_n + y_n|^{p-1}$$

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Theorem 1.2.4 (continued)

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$$\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q}$$

$$+ \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)}\right)^{1/q},$$

and since $q(p-1) = p$,

$$\sum_{n=1}^{\infty} |x_n+y_n|^p \leq \left\{ \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \right\} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/q}.$$

Since $1 - 1/q = 1/p$, Minkowski's Inequality follows. □

Theorem 1.2.4 (continued)

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$$\leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q}$$

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Corollary 1.2.A

Corollary 1.2.A. The ℓ^p spaces for $p \geq 1$ are vector spaces.

Proof. As commented above, we need only show these spaces are closed under scalar multiplication and vector addition. First, for any $\{x_n\}, \{y_n\} \in \ell^p$, and $\lambda \in \mathbb{C}$ we have that $\lambda\{x_n\} = \{\lambda x_n\}$ satisfies $\sum_{n=1}^{\infty} |\lambda x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore, $\lambda\{x_n\} \in \ell^p$.

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$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}$$

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