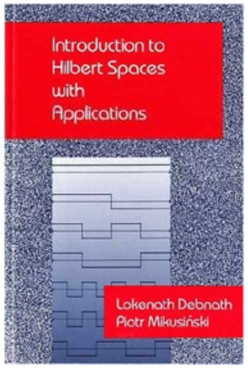
Advanced Differential Equations

Chapter 1. Normed Vector Spaces

Section 1.4. Normed Spaces—Proofs of Theorems



Theorem 1.4.1

Theorem 1.4.1. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms in a vector space E. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there exist positive α and β such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \qquad \text{(a)}$$

for all $x \in E$.

Proof. First if $x_n \to 0$ under either $\|\cdot\|_1$ or $\|\cdot\|_2$, then hypothesis (a) implies $x_n \to 0$ under $\|\cdot\|_2$ and $\|\cdot\|_1$ respectively.

Second, assume he norms are equivalent. ASSUME there is no $\alpha > 0$ such that $\alpha\|x\|_1 \leq \|x\|_2$ for all $x \in E$. Then for all $n \in \mathbb{N}$ there exists $x_n \in E$ such that $\frac{1}{n}\|x_n\|_1 > \|x_n\|_2$. Let $y_n = \frac{1}{\sqrt{n}}\frac{x_n}{\|x_n\|_2}$. Then $\|y_n\|_2 = 1/\sqrt{n} \to 0$. However, $\|y\|_1 > n\|y_n\|_2 = \sqrt{n}$. But then $y_n \to 0$ under $\|\cdot\|_2$ and $\|y_n\|_1 \to \infty$, CONTRADICTING the assumed equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$. Therefore such an α exists. Similarly, the required β exists and hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Theorem 1.4.6

Theorem 1.4.6. Compact sets are closed and bounded (in general).

Proof. Let S be a compact subset of E. Suppose $\{x_n\} \subset S$ and $x_n \to x$. Then by hypothesis there exists $\{x_{p_n}\} \subset \{x_n\}$ which converges to some $y \in S$. But $x_n \to x$, so $x = y$ and $x_n \to x \in S$. Therefore S is closed by Theorem 1.4.3.

Next, ASSUME S is not bounded. Then there exists a sequence $\{x_n\} \subset S$ such that $\|x_n\| \geq n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ contains no convergent subsequence and so S is not compact. Therefore if S is compact, then S is bounded. □