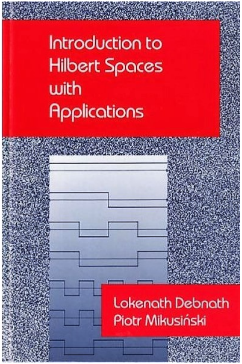
Advanced Differential Equations

Chapter 1. Normed Vector Spaces

Section 1.4. Normed Spaces—Proofs of Theorems



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for all $x \in E$.

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Second, assume he norms are equivalent. ASSUME there is no $\alpha > 0$ such that $\alpha\|x\|_1 \leq \|x\|_2$ for all $x \in E$. Then for all $n \in \mathbb{N}$ there exists $x_n \in E$ such that $\frac{1}{n}\|x_n\|_1 > \|x_n\|_2$. Let $y_n = \frac{1}{\sqrt{n}}\frac{x_n}{\|x_n\|_2}$. Then $\|y_n\|_2 = 1/\sqrt{n} \to 0$.

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Theorem 1.4.6. Compact sets are closed and bounded (in general).

Proof. Let S be a compact subset of E. Suppose $\{x_n\} \subset S$ and $x_n \to x$. Then by hypothesis there exists $\{x_{p_n}\} \subset \{x_n\}$ which converges to some $y \in S$. But $x_n \to x$, so $x = y$ and $x_n \to x \in S$. Therefore S is closed by Theorem 1.4.3.

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