Advanced Differential Equations

Chapter 1. Normed Vector Spaces

Section 1.5. Banach Spaces—Proofs of Theorems

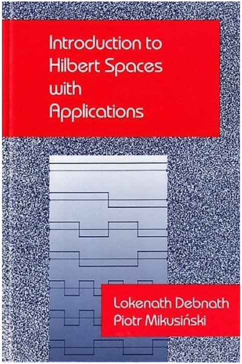

Table of contents

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$$\|x_n - x_m\| = \|(x_n - x) + (x - x_n)\| \leq \|x_n - x\| + \|x_m - n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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Theorem 1.5.A/Example 1.5.1

Theorem 1.5.A/Example 1.5.1. The space ℓ^2 is a Banach space.

Proof. Let $a_n = \{\alpha_{n,1}, \alpha_{n,2}, \ldots\}$ for $n \in \mathbb{N}$ be a Cauchy sequence in ℓ^2. We need to show a_n converges in ℓ^2.

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Given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m, n > M$

$$\|a_n - a_m\|^2 = \sum_{k=1}^{\infty} |\alpha_{m,k} - \alpha n, k|^2 < \varepsilon. \qquad (1.5.1)$$

Hence $|\alpha_{m,k} - \alpha_{n,k}|^2 < \varepsilon$ for $k \in \mathbb{N}$. Therefore for any fixed k^*, $\{\alpha_{n,k^*}\}$ is a Cauchy sequence in $\mathbb{C}$ and so is convergent. Say $\lim_{n\to\infty} \alpha_{n,k^*} = \alpha_{k^*} = \alpha_k$ and denote $a = \{\alpha_{1^*}, \alpha_{2^*}, \ldots\}$. We will show $a \in \ell^2$ and $a_n \to a$.

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Now (1.5.1) implies (see Exercise 1.15)

$$\sum_{k=1}^{k_0} (|\alpha_{m,k}| - |\alpha_{n,k}|)^2 \leq \sum_{k=1}^{k_0} |\alpha_{m,k} = \alpha_{n,k}|^2 < \varepsilon \text{ for all } k_0.$$

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Theorem 1.5.A/Example 1.5.1 (continued 1)

Proof (continued). Letting $m \to \infty$ yields

$$\sum_{k=1}^{k_0} (|\alpha_k| - |\alpha_{n,k}|)^2 \le \varepsilon. \qquad (1.5.2)$$

Now with $k_0 \to \infty$ in (1.5.2) we get

$$\sum_{k=1}^{\infty} (|\alpha_k| - |\alpha_{n,k}|)^2 \le \varepsilon. \qquad (1.5.3)$$

Since $\sum_{k=1}^{\infty} |\alpha_{n,k}|^2 < \infty$ we have by Minkowski's Inequality

$$\|a\| = \sqrt{\sum_{k=1}^{\infty} |\alpha_k|^2} = \sqrt{\sum_{k=1}^{\infty} (|\alpha_k| - |\alpha_{n,k}| + |\alpha_{n,k}|)^2}$$

$$\le \sqrt{\sum_{k=1}^{\infty} (|\alpha_k| - |\alpha_{n,k}|)^2} + \sqrt{\sum_{k=1}^{\infty} |\alpha_{n,k}|^2} < \infty.$$

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Proof (continued). Therefore $a \in \ell^2$.

Next, from (1.5.3), since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \sqrt{\sum_{k=1}^{\infty} (|\alpha_k| - |\alpha_{n,k}|)^2} = 0,$$

that is $\lim_{n\to\infty} \|a - a_n\| = 0$ and so $\lim_{n\to\infty} a_n = a$. □

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Theorem 1.5.2

Theorem 1.5.2. A normed space is complete if and only if every absolutely convergent series converges.

Proof. First, let E be a Banach space (and therefore complete). Let x_n be a sequence in E where $\sum_{n=1}^{\infty} \|x_n\| < \infty$ (i.e., x_n is absolutely convergent). Define $s_n = \sum_{k=1}^{n} x_k$. We will show that s_n is Cauchy and therefore the series $\sum_{n=1}^{\infty} x - n$ is convergent.

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$$\|s_m - s_n\| = \|x_{n+1} + x_{n+2} + \cdots + x_m\| \leq \sum_{r=n+1}^{\infty} \|x_r\| < \varepsilon.$$

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