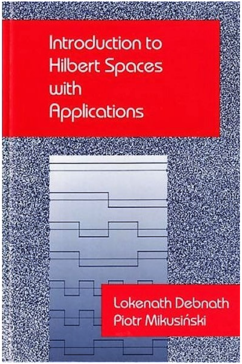
Advanced Differential Equations

Chapter 1. Normed Vector Spaces

Section 1.6. Linear Mappings—Proofs of Theorems



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Theorem 1.6.2

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Proof. Of course if L is continuous, it is continuous at a point. Now suppose f is continuous at a point. Let $x \in E$ and let $\{x_n\}$ be a sequence with $x_n \to x$. Then the sequence $\{x_n - x + x_0\}$ converges to x_0 and

$$\|L(x_n) - L(x)\| = \|L(x_n - x + x_0) - L(x_0)\| \to 0$$

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Theorem 1.6.3

Theorem 1.6.3. A linear mapping is continuous if and only if it is bounded.

Proof. First, suppose L is bounded and let $x_n \to 0$. Then $\|L(x_n)\| \leq K\|x_n\| \to 0$. Therefore L is continuous at $0 \in E_1$, and by Theorem 1.6.2, L is continuous.

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Second, suppose L is NOT bounded. Then for all $n \in \mathbb{N}$ there exists $x_n \in E_1$ such that $\|L(x_n)\| > n\|x_n\|$. Define $y_n = x_n/(n\|x_n\|)$ for $n \in \mathbb{N}$. Then $y_n \to 0$, but

$$\|L(y_n)\| = \left\| L\left(\frac{x_n}{x\|x_n\|}\right)\right\| = \left\|\frac{L(x_n)}{n\|x_n\|}\right\| > 1.$$

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Theorem 1.6.4

Theorem 1.6.4. If E_1 and E_2 are normed spaces then $\mathcal{B}(E_1, E_2)$ is a normed space with norm as given above.

Proof. "Clearly" the first two properties of a norm are satisfied. We need to demonstrate the Triangle Inequality. Let $L_1, L_2 \in \mathcal{B}(E_1, E_2)$. Then for all $x \in E_1$ such that $\|x\| = 1$ we have

$$\|L_1(x) - L_2(x)\| \leq \|L_1(x)\| + \|L_2(x)\|$$

(by the Triangle Inequality in E_2).

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$$\|L_1(x) + L_2(x)\| \leq \sup_{\|x\|=1} \|L_1(x)\| + \sup_{\|x\|=1} \|L_2(x)\| = \|L_1\| + \|L_2\|.$$

Therefore

$$\|L_1 + L_2\| = \sup_{\|x\|=1} \|L_1(x) + L_2(x)\| \leq \|L_1\| + \|L_2\|.$$

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Theorem 1.6.5. If E_1 is a normed space and E_2 is a Banach space, then $\mathcal{B}(E_1, E_2)$ is a Banach space.

Proof. We need to show completeness. Let $\{L_n\}$ be a Cauchy sequence in $\mathcal{B}(E_1, E_2)$ and let $x \in E_1$. Then $\|L_m(x) - L_n(x)\| \leq \|L_m - L_n\| \, \|x\| \to 0$ as $m, n \to \infty$. Therefore $\{L_n(x)\} \subset E_2$ is a Cauchy sequence and so $L_n(x) \to y \in E_2$. Define a mapping $L : E_1 \to E_2$ such that for this $x \in E_1$, $L(x) = y = \lim_{n\to\infty} L_n(x)$.

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Since each L_n is linear, L is linear. Since Cauchy sequences are bounded, we have

$$\|L(x)\| = \left\| \lim_{n\to\infty} L_n(x) \right\| = \lim_{n\to\infty} \|L_n(x)\| \le (\sup_{n\in\mathbb{N}} \|L_n\|)\|x\| < \infty.$$

So L is bounded and $L \in \mathcal{B}(E_1, E_2)$. We need to show (in the functional norm) $\|L_n - L\| \to 0$.

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Proof (continued). Let $\varepsilon > 0$ and let $k \in \mathbb{R}$ such that for all $m, n \geq k$ we have $\|L_m - L_n\| < \varepsilon$. If $\|x\| = 1$ and $m, n \geq k$ then $\|L_m(x) - L_n(x)\| \leq \|L_m - L_m\| < \varepsilon$. With $n \to \infty$, we see $\|L_m(x) - L(x)\| \leq \varepsilon$ for $m > k$ and x such that $\|x\| = 1$. Therefore $\{L_m(x)\} \to L(x)$. □