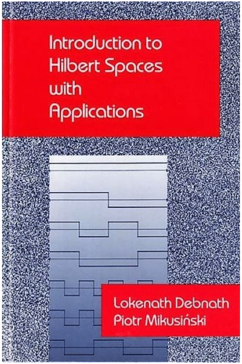
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.11. Linear Functionals and the Riesz Representation Theorem—Proofs of Theorems



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Let f be a bounded linear functional on a Hilbert space H. There exists exactly one $x_0 \in H$ such that $f(x) = (x, x_0)$ for all $x \in H$. Also $\|f\| = \|x_0\|$.

Proof. If $f(x) = 0$ for all $x \in H$, then take $x_0 = 0$. So without loss of generality $f(z) \neq 0$ for some $\in H$. Let $F = \{x \in H \mid f(x) = 0\}$. Since f is linear and bounded, it is continuous and so F is a (topologically) closed subspace of H.

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$$f\left(x - \frac{f(x)z_1}{f(z_1)}\right) = f(x) - \frac{f(x)f(z_1)}{f(z_1)} = 0.$$

And hence $(x - f(x)z_1/f(z_1), z_1) = 0$ since $z_1 \in F^\perp$.

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Proof (continued). Consequently

$$f(x)\left(\frac{z_1}{f(z_1)}, z_1\right) = (z, z_1).$$

Therefore with $x + 0 = \overline{f(z_1)} z_1/(z_1, z_1)$, then $f(x) = (x, x_0)$ for all $x \in H$.

Next suppose there is another x_1 such that $f(x) = (x, x_1)$ for all $x \in H$. Then $(x, x_0 - x_1) = 0$ for all $x \in H$ and so with $x = x_0 - x_1$, we have $(x_0 - x_1, x_0 - x_1) = 0$ and so $x_0 = x_1$.

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Proof (continued). Finally we have

$$\begin{aligned} \|f\| &= \sup_{\|x\|=1} |f(x)| = \sup_{\|x\|=1} |(x, x_0)| \\ &\leq \sup_{\|x\|=1} \|x\| \, \|x_0\| \text{ Schwarz's Inequality (Theorem 3.4.1)} \\ &= \|x_0\|. \end{aligned}$$

Alternatively,

$$\|x_0\|^2 = (x_0, x_0) = |f(x_0)| \leq \|f\| \, \|x_0\|.$$

Therefore $\|f\| = \|x_0\|$. $\square$