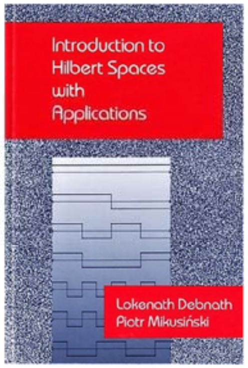
Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems
Section 3.12. Separable Hilbert Spaces—Proofs of Theorems



Theorem 3.12.1

Theorem 3.12.1. Every separable Hilbert space contains a countable dense subset.

Proof. Let $\{x_n\}$ be a complete orthonormal sequence in Hilbert space H. Let

$$S = \{(\alpha_1+\beta_1)x_1 + (\alpha_2+\beta_2 i)x_2 + \cdots + (\alpha_n+\beta_n i)x_n \mid \alpha_i, \beta_i \in \mathbb{Q}, n \in \mathbb{N}\}.$$

Then S is dense in H and countable. □

Theorem 3.12.2

Theorem 3.12.2. Every set of mutually orthogonal vectors in a separable Hilbert space is countable.

Proof. Let S be a set of mutually orthogonal vectors in a separable Hilbert space, and let $S_1 = \{x/\|x\| \mid x \in S, x \neq 0\}$. For any $x, y \in S_1$ we have $\|x-y\|^2 = 2$. Consider the set of open sets $\mathcal{B} = \{B(x, \sqrt{2}/2) \mid x \in S_1\}$. This is a collection of disjoint balls in the Hilbert space. By Theorem 3.12.1, the space contains a countable dense set D. Since the set is dense, each ball in $\mathcal{B}$ contains at least one element of D and since the balls are disjoint, different balls contain different elements of D. So there is a one to one mapping of $\mathcal{B}$ into D. Therefore $\mathcal{B}$ is countable and sets S_1 and S are countable. □

Theorem 3.12.3. Fundamental Theorem of Infinite Dimensional Vector Spaces

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Let H be a separable Hilbert space over scalar field $\mathbb{C}$. Then:

(b) if H is infinite dimensional then H is isomorphic to ℓ^2.

Proof. Let $\{x_n\}$ be a complete orthonormal sequence in H. If H is infinite dimensional then $\{x_n\}$ is an infinite sequence. Let x be an element of H. Define $T(x) = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_n = (x, x_n)$ for $n \in \mathbb{N}$. By Theorem 3.8.3, T is one to one mapping from H onto ℓ^2. Also, T is linear. We need only show that T preserves inner products. Denote $\alpha_n = (x, x_n)$ and $\beta = (y, x)n)$ for $x, y \in H$.

Theorem 3.12.3 (continued)

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Proof.
Then

$$(T(x), T(y)) = ((\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots)) = \sum_{n=1}^{\infty} \alpha_n \overline{\beta}_n = \sum_{n=1}^{\infty} (x, x_0)\overline{(y, x_0)}$$

$$= \sum_{n=1}^{\infty} (x, (y, x_n)x_n) = \left(x, \sum_{n=1}^{\infty} (y, x_n)x_n \right) = (x, y)$$

since the inner product is continuous. □