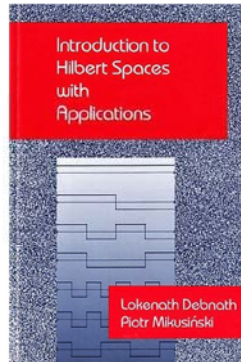


Advanced Differential Equations

Chapter 3. Hilbert Spaces and Orthonormal Systems

Section 3.4. Norm in an Inner Product Space—Proofs of Theorems



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Advanced Differential Equations

April 21, 2019

1 / 7

Theorem 3.4.1. Schwarz's Inequality

Theorem 3.4.1 (continued)

Theorem 3.4.1. Schwarz's Inequality.

For any x and y in an inner product space $|(x, y)| \leq \|x\| \|y\|$. Equality holds if and only if x and y are linearly dependent.

Proof (continued). Next, if x and y are linearly dependent then $y = \alpha x$ and the inequality reduces to equality. Conversely, suppose $|(x, y)| = \|x\| \|y\|$ or equivalently

$$(x, y)(y, x) = (x, x)(y, y). \quad (*)$$

Then by (*),

$$\begin{aligned} & ((y, y)x - (x, y)y, (y, y)x - (x, y)y) \\ &= (y, y)^2(x, x) - (y, y)(y, x)(x, y) - (x, y)(y, y)(y, x) + (x, y)(y, x)(y, y) = 0. \end{aligned}$$

Therefore $(y, y)x - (x, y)y = 0$ and x and y are linearly dependent. \square

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Advanced Differential Equations

April 21, 2019

4 / 7

Theorem 3.4.1. Schwarz's Inequality

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Theorem 3.4.1. Schwarz's Inequality.

For any x and y in an inner product space $|(x, y)| \leq \|x\| \|y\|$. Equality holds if and only if x and y are linearly dependent.

Proof. If $y = 0$, the inequality is satisfied. So without loss of generality, suppose $y \neq 0$. We know

$$0 \leq (x + \alpha y, x + \alpha y) = (x, x) + \overline{\alpha}(x, y) + \alpha(y, x) + |\alpha|^2(y, y).$$

Let $\alpha = -(x, y)/(y, y)$. Then

$$\begin{aligned} 0 &\leq (x, x) - \overline{\left(\frac{(x, y)}{(y, y)}\right)}(x, y) - \frac{(x, y)}{(y, y)}(y, x) + \left|\frac{(x, y)}{(y, y)}\right|^2(y, y) \\ &= (x, x) - \frac{|(x, y)|^2}{(y, y)} - \frac{|(x, y)|^2}{(y, y)} + \frac{|(x, y)|^2}{(y, y)} = (x, x) - \frac{|(x, y)|^2}{(y, y)} \end{aligned}$$

or $0 \leq (x, x)(y, y) - |(x, y)|^2$ or $|(x, y)| \leq \|x\| \|y\|$.

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Advanced Differential Equations

April 21, 2019

3 / 7

Corollary 3.4.1

Corollary 3.4.1

Corollary 3.4.1. For any two elements x and y of an inner product space we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof. We have

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + \overline{(x, y)} + (y, y) \\ &= (x, x) + 2\operatorname{Re}((x, y)) + (y, y) \\ &\leq (x, x) + 2|(x, y)| + (y, y) \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \text{ by Schwarz's Inequality (Theorem 3.4.1)} \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

\square

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Advanced Differential Equations

April 21, 2019

5 / 7

Theorem 3.4.2. Parallelogram Law

Theorem 3.4.2. Parallelogram Law.

For any two elements x and y of an inner product space

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. We have

$$\begin{aligned}\|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) \\ &= \|x\|^2 + (x, y) + (y, x) + \|y\|^2. \quad (*)\end{aligned}$$

Replacing y with $-y$:

$$\|x - y\|^2 = \|x\|^2 - (x, y) - (y, x) + \|y\|^2. \quad (**)$$

Adding $(*)$ and $(**)$, the result follows. \square

Theorem 3.4.3. Pythagorean Theorem

Theorem 3.4.3. Pythagorean Theorem.

If x and y are orthogonal then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. We have

$$\|x + y\|^2 = (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) = \|x\|^2 + \|y\|^2.$$

\square